BMS

INSTITUTE OF TECHNOLOGY AND MANAGEMENT

Avalahalli, Doddaballapur Main Road, Bengaluru – 560064

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

COURSE: INFORMATION THEORY AND CODING

COURSE CODE: 18EC54

SEMESTER: V

Index

B. E. (EC / TC)
Choice Based Credit System (CBCS) and Outcome Based Education (OBE)
SEMESTER – V

INFORMATION THEORY and CODING

Course Code	**18EC54**	**CIE Marks**	**40**
Number of Lecture Hours/Week	**3**	**SEE Marks**	**60**
Total Number of Lecture Hours	**40 (8 Hours / Module)**	**Exam Hours**	**03**

CREDITS – 03

Course Learning Objectives: This course will enable students to

- Understand the concept of Entropy, Rate of information and order of the source with reference to dependent and independent source.
- Study various source encoding algorithms.
- Model discrete & continuous communication channels.
- Study various error control coding algorithms.

Module-1	**RBT Level**
Information Theory: Introduction, Measure of information, Information content of message, Average Information content of symbols in Long Independent sequences, Average Information content of symbols in Long dependent sequences, Markov Statistical Model for Information Sources, Entropy and Information rate of Markoff Sources **(Section 4.1, 4.2 of Text 1)**	**L1, L2,L3**
Module-2	
Source Coding: Encoding of the Source Output, Shannon's Encoding Algorithm**(Sections 4.3, 4.3.1 of Text 1),** Shannon Fano Encoding Algorithm **(Section 2.15 of Reference Book 4)** Source coding theorem, Prefix Codes, Kraft McMillan Inequality property – KMI, Huffman codes **(Section 2.2 of Text 2)**	**L1, L2,L3**
Module-3	
Information Channels: Communication Channels, Discrete Communication channels Channel Matrix, Joint probabilty Matrix, Binary Symmetric Channel, System Entropies. **(Section 4.4, 4.5, 4.51,4.5.2 of Text 1)** Mutual Information, Channel Capacity, Channel Capacity of Binary Symmetric Channel, **(Section 2.5, 2.6 of Text 2)** Binary Erasure Channel, Muroga,s Theorem **(Section 2.27, 2.28 of Reference Book 4)**	**L1, L2, L3**
Module-4	
Error Control Coding: Introduction, Examples of Error control coding, methods of Controlling Errors, Types of Errors, types of Codes, Linear Block Codes: matrix description of Linear Block Codes, Error detection & Correction capabilities of Linear Block Codes, Single error correction Hamming code, Table lookup Decoding using Standard Array. **Binary Cyclic Codes:** Algebraic Structure of Cyclic Codes, Encoding using an (n-k) Bit Shift register, Syndrome Calculation, Error Detection and Correction **(Sections 9.1, 9.2,9.3,9.3.1,9.3.2,9.3.3 of Text 1)**	**L1, L2, L3**
Module-5	
Convolution Codes: Convolution Encoder, Time domain approach, Transform domain approach, Code Tree, Trellis and State Diagram, The Viterbi Algorithm) **(Section 8.5 – Articles 1,2 and 3, 8.6- Article 1 of Text 2)**	**L1, L2, L3**

Course Outcomes: After studying this course, students will be able to:

- Explain concept of Dependent & Independent Source, measure of information, Entropy, Rate of Information and Order of a source
- Represent the information using Shannon Encoding, Shannon Fano, Prefix and Huffman Encoding Algorithms
- Model the continuous and discrete communication channels using input, output and joint probabilities
- Determine a codeword comprising of the check bits computed using Linear Block codes, cyclic codes & convolutional codes

- Design the encoding and decoding circuits for Linear Block codes, cyclic codes, convolutional codes, BCH and Golay codes.

Question paper pattern:

- Examination will be conducted for 100 marks with question paper containing 10 full questions, each of 20 marks.
- Each full question can have a maximum of 4 sub questions.
- There will be 2 full questions from each module covering all the topics of the module.
- Students will have to answer 5 full questions, selecting one full question from each module.
- The total marks will be proportionally reduced to 60 marks as SEE marks is 60.

Text Book:

1. Digital and analog communication systems, K. Sam Shanmugam, John Wiley India Pvt. Ltd, 1996.
2. Digital communication, Simon Haykin, John Wiley India Pvt. Ltd, 2008.

Reference Books:

1. ITC and Cryptography, Ranjan Bose, TMH, II edition, 2007
2. Principles of digital communication, J. Das, S. K. Mullick, P. K. Chatterjee, Wiley, 1986 - Technology & Engineering
3. Digital Communications – Fundamentals and Applications, Bernard Sklar, Second Edition, Pearson Education, 2016, ISBN: 9780134724058.
4. Information Theory and Coding, HariBhat, Ganesh Rao, Cengage, 2017.
5. Error Correction Coding by Todd K Moon,Wiley Std. Edition, 2006

29/7/19

INFORMATION THEORY AND CODING

17EC54 Ambika Mam

Information : Probability of occurrence
↳ amount of message / surprise

$I \propto \frac{1}{P}$

$I = \log \frac{1}{P}$
↓
proportionality constant

$I = \log_2 \frac{1}{P}$ bits ⇒ frequently used

$I = \log_{10} \frac{1}{P}$ Hartleys

$I = \log_e \frac{1}{P}$ Nats

Module - I : Information Theory

$I \propto \frac{1}{P}$; $I = \log \frac{1}{P}$

Information that we get or conveyed is always positive. If we don't convey any information then information is 0.

* Why log is used as proportionality constant?

1. Information cannot be negative.
2. The lowest possible self information is 0.
3. More information is carried by less likely message.
4. If the no. of informations are more, the total information is the sum of all individual informations.

30/7/19

$S = S_k$ and S_l ⇒ Information Symbols

⇒ P_k P_l → Probabilities

I = (Information of S_k) and (Information of S_l)

The only operation that converts 'and' into addition is 'log'. So log is the proportionality constant.

Proof:-

Two independent symbols S_k and S_l are transmitted with probabilities P_k and P_l respectively. Then, the total self information is given by

$$I_{kl} = \log \frac{1}{(P_k \text{ and } P_l)} = \log \frac{1}{(P_k \cap P_l)}$$

$$= \log \frac{1}{P(S_k) P(S_l)} = \log \frac{1}{P_k . P_l}$$

$$= \log \frac{1}{P_k} + \log \frac{1}{P_l}$$

$$\boxed{I_{kl} = I_k + I_l}$$

Zero memory source or Independent source

Probability of occurrence of any event is independent of probability of occurrence of previous event is called as zero memory source.

Average Information Source content [Entropy] of symbols in long independent sequences

30/7/

Symbol is a combination of bits.

$$S = \{ S_k, S_l, S_m, S_n, S_q \}$$

$$\underset{4\text{bits}}{\downarrow}\ \underset{8\text{bits}}{\downarrow}\ \underset{2}{\downarrow}\ \underset{10}{\downarrow}\ \underset{20}{\downarrow} \rightarrow \text{length of symbols}$$

Derivation

Let us consider zero memory source producing independent sequences of symbols. The receiver of these sequences may interpret the entire message a a single unit.

Let us consider the source alphabet

$S = \{S_1, S_2 \dots S_q\}$ with probabilities

$P = \{P_1, P_2 \dots P_q\}$ respectively.

Let us consider long independent sequence of length "L" symbols. This long sequence then contains

$P_1 L$ no. of messages of type S_1

$P_2 L$ no. of messages of type S_2

⋮

$P_q L$ no. of messages of type S_q.

Self information of $S_1 = \log \frac{1}{P_1}$ bits

$\therefore$ $P_1 L$ no. of messages of type S_1 contain $P_1 L \log \frac{1}{P_1}$ bits of information.

III^{ly} $P_2 L$ no. of messages of type S_2 contain $P_2 L \log \frac{1}{P_2}$ bits of information.

⋮

$P_q L$ no. of messages of type S_q contain $P_q L \log \frac{1}{P_q}$ bits of information.

30/7/19

$$I_{total} = P_1 L \log \frac{1}{P_1} + P_2 L \log \frac{1}{P_2} + - - - - + P_q L \log \frac{1}{P_q}$$

$$I_{total} = L \sum_{i=1}^{q} P_i \log \frac{1}{P_i}$$

∴ The average self information = $\frac{I_{total}}{L}$

$$\boxed{H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i}}$$

[S → source
If we represent source as A then Avg. inf^m will be H(A)]

<u>Problems</u> To understand meaning of entropy

① Let us consider a binary source with source alphabet $S = \{S_1, S_2\}$ with probabilities $P = \left\{\frac{1}{256}, \frac{255}{256}\right\}$. Find entropy.

<u>Sol^n</u>:-

$$H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} = \sum_{i=1}^{2} P_i \log \frac{1}{P_i}$$

$$= P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}$$

$$= \frac{1}{256} \log_2 \frac{1}{\frac{1}{256}} + \frac{255}{256} \log \frac{1}{\frac{255}{256}}$$

$$= \frac{1}{256} \frac{\log_{10} 256}{\log_{10} 2} + \frac{255}{256} \frac{\log_{10} \frac{256}{255}}{\log_{10} 2}$$

$$= 0.031 + 0.0056$$

$$H(S) = 0.0369 \text{ bits / Message symbol}$$

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② $S' = \{S_3, S_4\}$ $P' = \left\{\frac{7}{16}, \frac{9}{16}\right\}$

$$H(S) = P_3 \log \frac{1}{P_3} + P_4 \log \frac{1}{P_4}$$

$$= \frac{7}{16} \frac{\log_{10} 16/7}{\log_{10} 2} + \frac{9}{16} \frac{\log_{10} \frac{16}{9}}{\log_{10} 2}$$

30/7/19

$= 0.5217 + 0.4669$

$= 0.9887$ bits / Message symbol

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③ $S'' = \{S_5, S_6\} \quad P'' = \{\frac{1}{2}, \frac{1}{2}\}$

$$H(S) = \sum_{i=5}^{6} P_i \log \frac{1}{P_i}$$

$$= P_5 \log \frac{1}{P_5} + P_6 \log \frac{1}{P_6}$$

$$= \frac{1}{2} \frac{\log_{10} 2}{\log_{10} 2} + \frac{1}{2} \frac{\log_{10} 2}{\log_{10} 2}$$

$H(S) = 1$ bit / Message symbol

If the entropy is very less then we can guess which event can occur next.

1 is the maximum entropy.

Entropy indicates whether we can guess the event that occurs next.

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④ Consider a source $S = \{S_1, S_2, S_3\}$ and $P = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

Find i) Self information of each message

ii) ~~Intrepret~~ the source S
Entropy of

Solⁿ :- i) Self information

$$I_1 = \log \frac{1}{P_1} = \frac{\log_{10} 2}{\log_{10} 2} = 1 \text{ bit}$$

$$I_2 = \log \frac{1}{P_2} = \log \frac{1}{1/4} = \frac{\log_{10} 4}{\log_{10} 2} = 2\text{bit}$$

$$I_3 = \log \frac{1}{P_3} = \log \frac{1}{1/4} = \frac{\log_{10} 4}{\log_{10} 2} = 2 \text{ bit}$$

ii) Entropy $H(S) = \sum_{i=1}^{3} P_i \log \frac{1}{P_i}$

$$= \frac{1}{2} \log \frac{1}{1/2} + 1/4 \log 1/1/4 + 1/4 \log 1/1/4$$

30/7/19

$$= \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{4}(2)$$

$$= 1.5 \text{ bits / message symbol}$$

Other problems

① The binary symbols 0 and 1 are transmitted wit probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively. Find the correspondin self informations.

Sol^n:- Check for the sum of all probabilities It should be $= 1$

Self information of emitting 0 $I = \log_2 \frac{1}{P} = \log_2 {}^{1}/_{1/4}$

$$= \frac{\log_{10} 4}{\log_{10} 2} = 2 \text{ bits}$$

Self information of emitting 1 $I = \log_2 {}^{1}/_{P} = \log_2 {}^{1}/_{3/4}$

$$= \frac{\log_{10} {}^{4}/_{3}}{\log_{10} 2} = 0.415 \text{ bits}$$

② Let us consider a binary source which emits the source alphabet $S = \{S_1, S_2\}$ with probability $\frac{1}{256}$ and $\frac{255}{256}$ respectively. Find the self information.

Sol^n:- Total $P = 1$

$$I = \log_2 \frac{1}{P} = \log_2 \frac{1}{{}^{1}/_{256}} = \frac{\log_{10} 256}{\log_{10} 2} = 8 \text{ bits}$$

$$I = \log_2 \frac{1}{P} = \log_2 \frac{1}{\frac{255}{256}} = \frac{\log_{10}\left(\frac{256}{255}\right)}{\log_{10} 2} = 0.0056 \text{ bits}$$

Formula $\log_b a = \frac{\log_{10} a}{\log_{10} b}$

Information rate R_s

1/8/

Rate at which the source is giving us the information is called information rate.

Let "r_s" symbols/sec be the information rate at which the source is giving us the information

"r_s" symbols/sec → message symbol rate

Average source information rate, $\boxed{R_s = H(s)\, r_s \text{ bits/se}}$

Problems

① A discrete source emits 1 of 6 symbols once every millisecond. Symbol probabilities are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ and $\frac{1}{32}$ respectively. Find the source entropy and the information rate.

Sol^n :- Entropy $H(S) = \sum_{i=1}^{6} P_i \log \frac{1}{P_i}$

$$= \frac{1}{2}\log_2 {}^1/_{1/2} + \frac{1}{4}\log_2 {}^1/_{1/4} + \frac{1}{8}\log_2 {}^1/_{1/8} + \frac{1}{16}\log_2 {}^1/_{1/16} + \frac{1}{32}\log_2 {}^1/_{1/32} + \frac{1}{32}\log_2 {}^1/_{1/32}$$

$$= \frac{1}{2} + \frac{1}{2} + 0.375 + 0.25 + 0.15625 + 0.15625$$

$H(s) = 1.9375$ bits/ message ~~signal~~ symbol

$r_s = 1 \;\; {}^1/_{10^{-3}}\, s = 1000$ [1 of 6 symbols/ms ∴ ${}^1/_{10^{-3}}$]

$R_s = H(S)\, r_s = 1.9375 \times 1000 =$ ~~11.625 bits/sec~~

$R_s = 1937.5$ bits/sec

② The output of an information source consists of 150 symbols, 32 of which occurred with probability $\frac{1}{64}$ and the remaining 108 occurred with probability $\frac{1}{236}$. The source emits 2000 symbols per second.

Assuming that symbols are choosen independently, find the average information rate of this source. 1/8/19

Sol^n:- $H(S) = \sum_{i=1}^{\overset{150}{\cancel{32}}} P_i \log \frac{1}{P_i}$

$$= \left(\frac{1}{64} \log_2 \frac{1}{1/64}\right) 32 + 108\left(\frac{1}{236} \log_2 \frac{1}{\frac{1}{236}}\right)$$

$$= 3 + \cancel{3.6073}\ 3.9413$$

$$H(S) = \cancel{6.6073}\ 6.9413 \text{ bits/message symbol}$$

$$\cancel{R(S)} = R_s = H(S)\, r_s$$

$$r_s = 2000$$

$$R_s = 2000 \times 6.9413$$

$$R_s = 13882.6 \text{ bits/sec}$$

Find the relationship between Hartleys, Nats and bits

$$I = \log_{10} \frac{1}{P} \text{ Hartleys} \quad — ①$$

$$I = \log_e \frac{1}{P} \text{ Nats} \quad — ②$$

$$I = \log_2 \frac{1}{P} \text{ bits} \quad — ③$$

From eq ①

$$1 \text{ Hartleys} = \frac{I}{\log_{10} \frac{1}{P}}$$

From eq ②

$$1 \text{ Hartleys} = \frac{\log_e \frac{1}{P}}{\log_{10} \frac{1}{P}}$$

$$= \frac{-\log_e P}{-\log_{10} P} \text{ Nats}$$

1/8/19

1 ~~Hartleys~~ = $\log_{p} e$

$$1 \text{ Hartleys} = \frac{\log_P 10}{\log_P e} \text{ Nats}$$

$$= \log_e 10 \text{ Nats}$$

$$\boxed{1 \text{ Hartleys} = 2.303 \text{ Nats}}$$

$$\boxed{\log_a b = \frac{1}{\log_b a}}$$

From eq ① $\quad 1 \text{ Hartleys} = \dfrac{I}{\log_{10} \frac{1}{P}}$

from eq ③ $\quad 1 \text{ Hartleys} = \dfrac{\log_2 \frac{1}{P}}{\log_{10} \frac{1}{P}}$

$$= \frac{-\log_2 P}{-\log_{10} P}$$

$$= \frac{\log_P 10}{\log_P 2}$$

$$= \log_2 10$$

$$\boxed{1 \text{ Hartleys} = 3.322 \text{ bits}}$$

From eq ② $\quad 1 \text{ Nats} = \dfrac{I}{\log_e \frac{1}{P}}$

from eq ③ $\quad 1 \text{ Nats} = \dfrac{\log_2 \frac{1}{P}}{\log_e \frac{1}{P}} = \dfrac{-\log_2 P}{-\log_e P}$

$$= \frac{\log_P e}{\log_P 2}$$

$$= \log_2 e$$

$$\boxed{1 \text{ Nats} = 1.443 \text{ bits}}$$

Problems

① The collector voltage of certain ckt is to lie between −5 and −12 V. The voltage can take only these values −5, −6, −7, −9, −11, −12 with respective probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}$. This voltage is recorded in a pen recorder. Determine the average self information associated with the record interms of bits per level.

Solⁿ:-

$$H(S) = \sum_{i=1}^{6} P_i \log \frac{1}{P_i}$$

$$= \frac{1}{6}\log_{10} 6 + \frac{1}{3}\log_{10} 3 + \frac{1}{12}\log_{10} 12 + \frac{1}{12}\log_{10} 12$$

$$+ \frac{1}{6}\log_{10} 6 + \frac{1}{6}\log_{10} 6$$

[1 Hartley = 3.32 bits]

$$= 0.7279 \times 3.32$$

$H(S) = 2.4168$ bits / message symbol

② A code is composed of dots and dashes. Assuming that dash is 3 times as long as a dot and has $\frac{1}{3}$rd the probability of occurrence, calculate i) the information in a dot and a dash. ii) The entropy of dot dash code. iii) The average rate of information if a dot lasts for 10 msec and this time is allowed between symbols.

Solⁿ:- Probability of dot $= \frac{1}{3}$

$P_{dot} + P_{dash} = 1$

$P_{dash} = \frac{1}{3} P_{dot}$

$$P_{dot} = \frac{3}{4}$$

Probability of dash $= \frac{1}{4}$

I of dot $= \log \frac{1}{P} = \log_{10} \frac{4}{3} = \overset{0.414}{\cancel{1.58}}$ bits ~~/symbol~~

I of dash $= \log \frac{1}{P} = \log_{10} 4 = 1.998$ bits

2/8/19

$P_{dot} + P_{dash} = 1$

$P_{dash} = \frac{1}{3} P_{dot}$

$P_{dot} + \frac{1}{3} P_{dot} = 1$

$3P_{dot} + P_{dot} = 3$

$4P_{dot} = 3$

$P_{dot} = \frac{3}{4}$

$P_{dash} = 1 - P_{dot}$

$= 1 - \frac{3}{4}$

$P_{dash} = \frac{1}{4}$

i) Self information of dot $= \log_2 \frac{1}{\frac{3}{4}} = \log_{10} \frac{4}{3} \times 3.32 = 0.4147$ bits

Self information of dash $= \log_2 \frac{1}{1/4} = \log_{10} 4 \times 3.32 = 1.998$ bits ≈ 2 bits

ii) Entropy $H(S) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i}$

$= \left(\frac{3}{4} \log_{10} \frac{4}{3} + \frac{1}{4} \log_{10} 4 \right) 3.32$

$= 0.8108$ bits/message symbol ≈ 0.811

iii) Average rate of information $R_S = H(S)\, r_s$

$t_{dot} = 10$ msec

$t_{dash} = 3 \times t_{dot} = 30$ msec

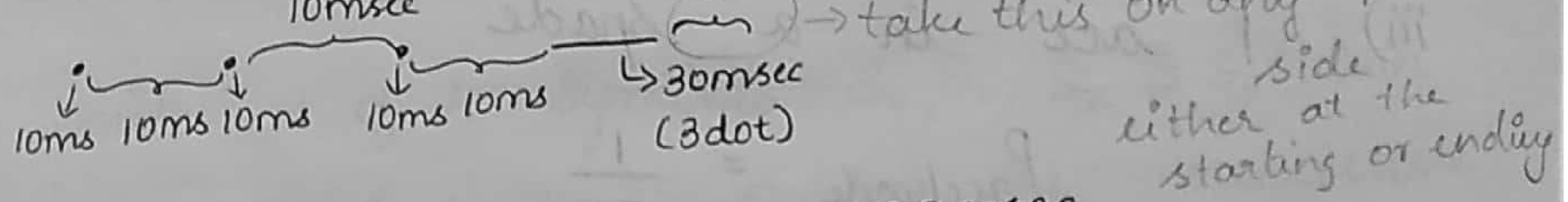
$t_{gap} = 10$ msec

t_{total} -

1 code consists of (3+1)=4 symbols 3 dots and 1 dash

The time gap between symbols is 10msec

Assume



$T_{total} = 10+10+10+10+10+10+30+10 = 100$ msec

$r_s = \frac{\cancel{100 msec}}{4}$ 4 symbols/100 msec

It is taking 100 msec to emit a code and that code consists of 4 symbols.

2/8/19

$R_s = H(S) : r_s$

$= (0.8113)(4 \text{ symbols}/100\text{ms})$

$= 0.0324 \times 10^3$

$R_s = 32.45$ bits/second

③ A card is drawn from a deck.

* i) You are told it is a spade. How much informatio did you receive?

ii) How much information did you receive if you are told that the card drawn is an ace?

iii) If you are told that the card drawn is an ace of spades, how much information did you receive?

iv) Is the information obtained in (iii) the sum of informations obtained in (i) and (ii)?

Sol^n:- There are 52 cards in deck

i) 13 → Spades

$$\text{Probability} = \frac{13}{52} = \frac{1}{4}$$

$$I_{sp} = \log_2 4 = 2 \text{ bits}$$

ii) 4 ace cards.

$$P_{ace} = \frac{4}{52} = \frac{1}{13}$$

$$I_{ace} = \log_2 13 = 3.7 \text{ bits}$$

iii) 1 ace in a spade

$$P_{ace|spade} = \frac{1}{52}$$

$$I_{a|s} = \log_2 52 = 5.7 \text{ bits}$$

iv) $I_{a|s} = I_{ace} + I_{sp}$

$5.7 = 2 + 3.7$

$5.7 = 5.7$

Yes the information obtained in (iii) is the sum of informations obtained in (i) and (ii).

2/8/19

Drawing an ace and drawing a spade are 2 mutually independent events and the total self information must be equal to individual self informations.

④ A source emits 1 of 4 possible symbols x_0 to x_3 during each signaling interval. The symbols occur with probabilities as given in table.

Symbols	Probabilities
x_0	0.4
x_1	0.3
x_2	0.2
x_3	0.1

Find the amount of information gain by observing the source emitting by each of these symbols and also the entropy of the source.

Sol^n:- Self information of each symbol

$$I_{x_0} = \log_2 \frac{1}{0.4} = 1.322 \text{ bits}$$

$$I_{x_1} = \log_2 \frac{1}{0.3} = 1.74 \text{ bits}$$

$$I_{x_2} = \log_2 \frac{1}{0.2} = 2.322 \text{ bits}$$

$$I_{x_3} = \log_2 \frac{1}{0.1} = 3.322 \text{ bits}$$

$$\text{Entropy of source} = \sum_{i=0}^{3} P_i \log \frac{1}{P_i}$$

$$= 0.4 \log_2 \frac{1}{0.4} + 0.3 \log_2 \frac{1}{0.3} + 0.2 \log_2 \frac{1}{0.2} + 0.1 \log_2 \frac{1}{0.1}$$

$H(S) =$ 1.8474 bits / message symbols

⑤ A source emits 1 of 4 symbols S_0, S_1, S_2 & S_3 with probabilities $\frac{1}{3}, \frac{1}{6}, \frac{1}{4}$ & $\frac{1}{4}$ respectively. The successive symbols emitted by source are statistically independent. Calculate the entropy of the source. Calculate the entropy in terms of

i) Nats / symbol
ii) bits / symbol
iii) Hartleys / symbol

2/8/19

Sol^n :- $H(S) = \sum_{i=1}^{4} P_i \log P_i$

$= \frac{1}{3}\log 3 + \frac{1}{6}\log 6 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4$

$H(S) = 0.5898$ Hartleys/symbol

$= 0.589 \times 2.303$ Nats/symbol

$H(S) = 1.358$ Nats/symbol

$= 0.5898 \times 3.2219$

$H(S) = 1.9592$ bits/symbol

⑥ A binary source is emitting an independent sequence of 0's and 1's with probabilities 'P' and (1-P) respectively. Plot the entropy of the source v/s probability.

Sol^n:- $H(S) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i}$

$P = 0.1 \qquad 1-P = 0.9$ or $P = 0.9 \quad 1-P = 0.1$

$H(S) = 0.1 \log_2 \frac{1}{0.1} + 0.9 \log_2 \frac{1}{0.9}$

$= 0.4689$ bits/message symbol

$P = 0.2 \qquad 1-P = 0.8$ or $P = 0.8 \quad 1-P = 0.2$

$H(S) = 0.2 \log_2 \frac{1}{0.2} + 0.8 \log_2 \frac{1}{0.8} = 0.7219$ bits/mes ~~signal~~ symbol

$P = 0.3 \qquad 1-P = 0.7$ or $P = 0.7 \quad 1-P = 0.3$

$H(S) = 0.3 \log_2 \frac{1}{0.3} + 0.7 \log_2 \frac{1}{0.7} = 0.8812$ bits/mes symbol

$P = 0.4 \qquad 1-P = 0.6$ or $P = 0.6 \quad 1-P = 0.4$

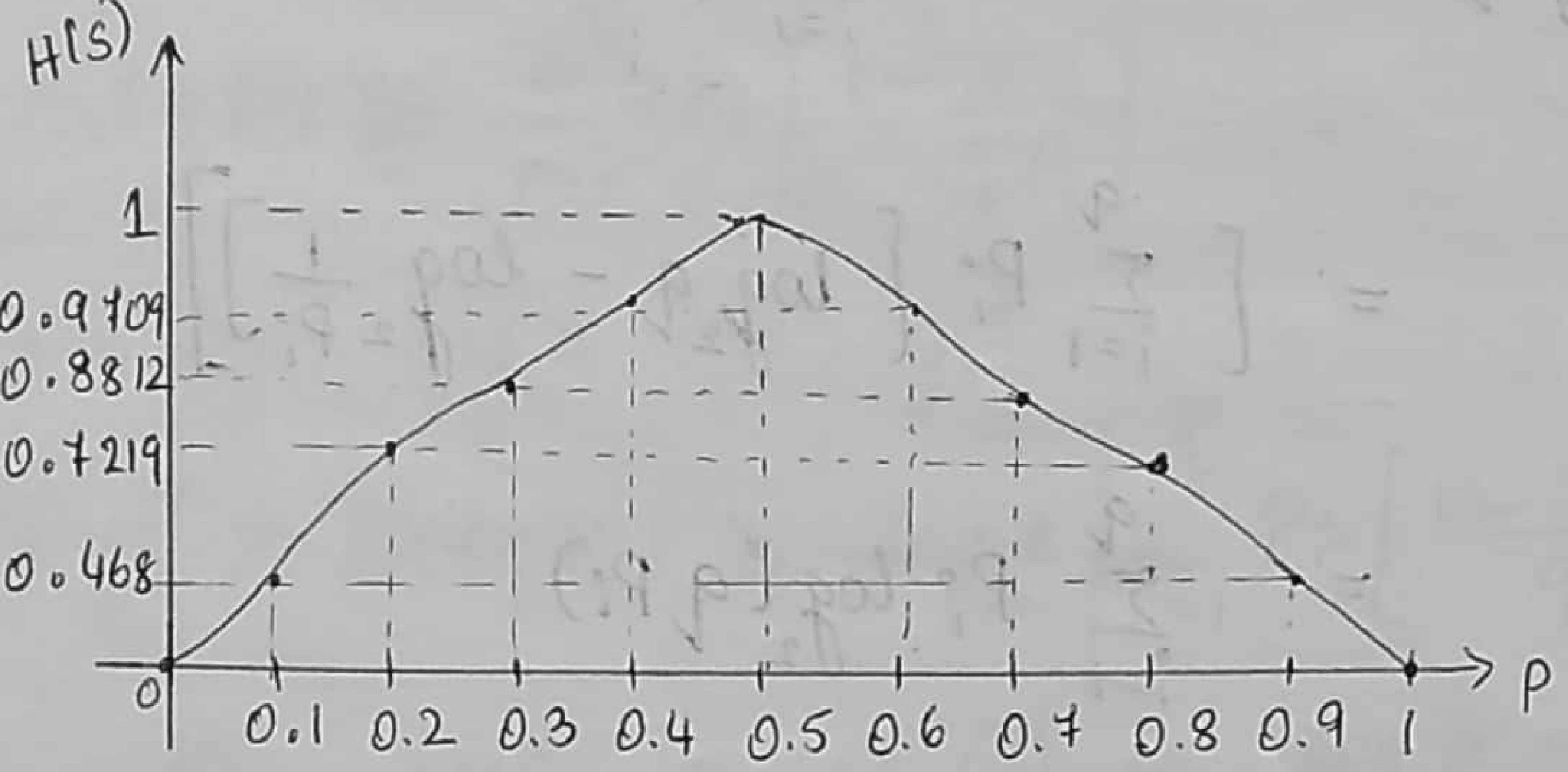
$H(S) = 0.4 \log_2 \frac{1}{0.4} + 0.6 \log_2 \frac{1}{0.6} = 0.9709$ bits/mes sym.

$P = 0.5$ $\quad 1-P = 0.5$ 2/8/19

$$H(S) = 0.5 \log_2 \frac{1}{0.5} + 0.5 \log_2 \frac{1}{0.5} = 1$$

$P = 0$ $\quad 1-P = 1$ or $\left[P = 1 \right]$ $\quad 1-P = 0$

$$H(S) = 1 \log_2 1 = 0$$



Entropy v/s probability

Properties of entropy

$$H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} = \sum_{i=1}^{q} P_i I_i$$

1. $P \rightarrow (0, 1)$ Avg information content is always positive
2. $H[P, (1-P)] = H[(1-P), P]$
3. Extremal property : To know the maximum value of entropy. Minimum value $= 0$ Continued.

* Derivation

Let us consider the source S with q symbols $S = (S_1, S_2, ---, S_q)$ with probabilities $P = \{P_1, P_2, -- P_q\}$ respectively.

Entropy $H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i}$

$$\sum_{i=1}^{q} P_i = 1 \quad — ①$$

Let us now prove that entropy H(S) has upper bound by considering the ~~probability~~ quantity $\log q - H(S)$

$$\log q - H(S) = \left[\sum_{i=1}^{q} P_i\right]\log_2 q - \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$$

$\log q = 1 \log q$ where $1 = \sum_{i=1}^{q} P_i$

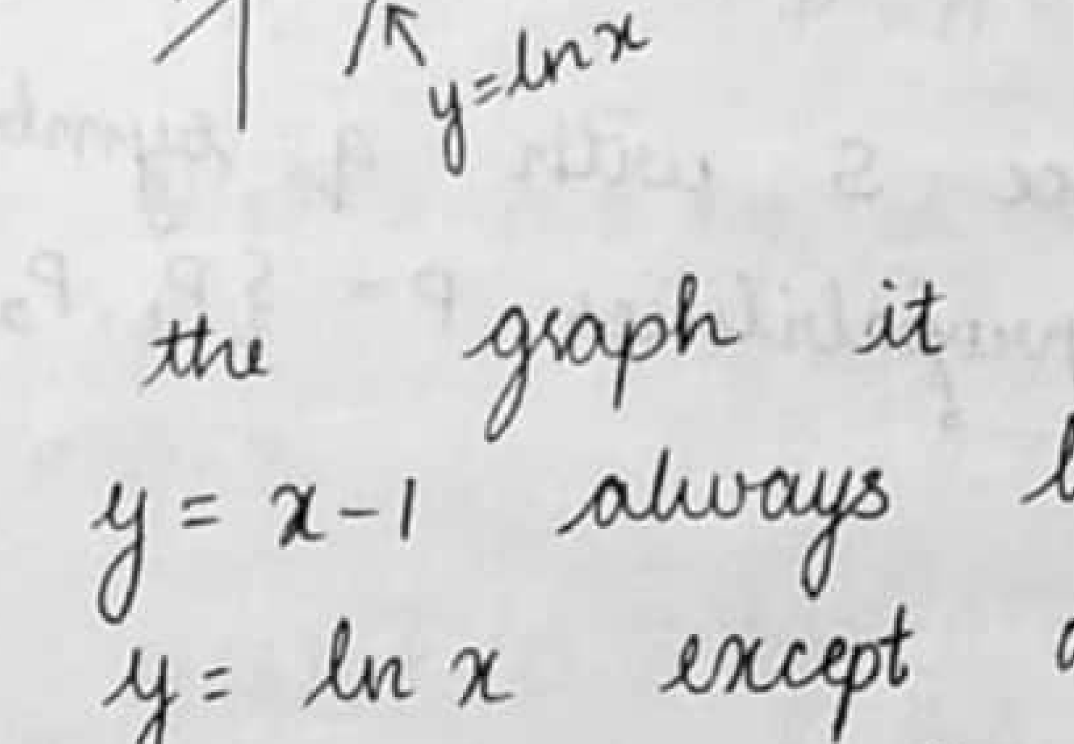
$$= \left[\sum_{i=1}^{q} P_i\left[\log_2 q - \log_2 \frac{1}{P_i}\right]\right]$$

$$= \sum_{i=1}^{q} P_i \log_2 (q P_i)$$

$$= \sum_{i=1}^{q} P_i \frac{\log_e q P_i}{\log_e 2}$$

$$\log q - H(S) = \frac{1}{\log_e 2} \sum_{i=1}^{q} P_i \log_e q P_i$$

$$\log q - H(S) = \log_2 e \sum_{i=1}^{q} P_i \log_e q P_i \quad - ② \quad = \log_2 e \sum_{i=1}^{q} P_i \ln q P_i$$



logaithmic curve is always lesser than straight line except at $x=1$. ∴ Straight line is acting as tangent.

From the graph it is evident that the straight line $y = x-1$ always lies above the logaithmic curve $y = \ln x$ except at $x=1$. Thus the straight line forms tangent to the curve at $x=1$. Therefore

$\ln x \leq$ ~~ts~~ $x-1$ $\quad x^{ly} -1$

$$-\ln x \geq 1-x \quad - ③$$

$\ln\left(\frac{1}{x}\right) \geq 1-x$

Let $x = \frac{1}{qP_i}$, $\ln(qP_i) \geq 1 - \frac{1}{qP_i}$ — ④

Multiplying both sides of eq ④ by P_i and then taking summation for all $i = 1, 2 \ldots q$ we get

$$\sum_{i=1}^{q} P_i \ln(qP_i) \geq \sum_{i=1}^{q} P_i \left[1 - \frac{1}{qP_i}\right]$$

x^ly b.s by $\log_2 e$ we get

$$\log_2 e \sum_{i=1}^{q} P_i \ln(qP_i) \geq \log_2 e \left[\sum_{i=1}^{q} P_i \left[1 - \frac{1}{qP_i}\right]\right] \text{ — ⑤}$$

From eq ②, LHS of eq ⑤ is $\log q - H(S)$

$$\log q - H(S) \geq \log_2 e \left[\sum_{i=1}^{q} P_i \left[1 - \frac{1}{qP_i}\right]\right]$$

By considering $P_i = \frac{1}{q}$ RHS of eq ⑤ becomes 0

$$\therefore \log q - H(S) \geq 0$$

or $\boxed{H(S) \leq \log q}$ — ⑥ $P_i = \frac{1}{q}$

H(s) is maximum when all the symbols ~~are~~ have equal probabilities.

Eg: If $q = 2$, $P_i = \frac{1}{q}$ means $P_1 = P_2 = \frac{1}{2}$

The equality sign holds good when $P_i - \frac{1}{q} = 0$ for all $i = 1, 2 \ldots q$ i.e $P_i = \frac{1}{q}$ for all $i = 1, 2 \ldots q$ — ⑦

When the condition of equation ⑦ is satisfied the entropy becomes maximum and is given by

$\boxed{H(S)_{max} = \log_2 q}$ bits/message symbol — ⑧

i.e, the entropy attains the maximum value when all the source symbols become equiprobable.

Illustrations

① Let us consider source emits 3 symbol $S = \{S_1, S_2, S_3\}$ with probabilities $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$. Find the entropy and maximum entropy.

Soln :- $H(S) = \sum_{i=1}^{3} P_i \log \frac{1}{P_i}$

$$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4$$

$$= \frac{1}{2} + \frac{2}{4} + \frac{2}{4}$$

$$\boxed{H(S) = 1.5 \text{ bits/message symbol}}$$

$H(S)_{max} = \log_2 q \qquad q = 3 \qquad H(S) = \log_2 3$

$H(S)_{max} = 1.585$ bits/message symbol

② 4 symbols $S = \{S_1, S_2, S_3, S_4\}$ $P = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\}$. Find $H(S)$ and $H(S)_{max}$

Soln :- $H(S) = \sum_{i=1}^{4} P_i \log \frac{1}{P_i}$

$$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8$$

$$= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{3}{8}$$

$H(S) = 1.75$ bits/message symbol

$H(S)_{max} = \log_2 q \qquad (q = 4) \quad = \log_2 4$

$H(S)_{max} = 2$ bits/message symbol

③ $q = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ find $H(S)_{max}$ for each q

Soln $\qquad q = 1 \qquad H(S)_{max} = \log_2 1 = 0$ bits/ms

$\qquad q = 2 \qquad H(S)_{max} = \log_2 2 = 1$ bits/ms

$q=3$ $\quad H(S)_{max} = \log_2 3 = 1.585$ bits / message symbol

$q=4$ $\quad H(S)_{max} = \log_2 4 = 2$ bits / message symbol

$q=5$ $\quad H(S)_{max} = \log_2 5 = 2.322$ bits / message symbol

$q=6$ $\quad H(S)_{max} = \log_2 6 = 2.585$ bits / message symbol

$q=7$ $\quad H(S)_{max} = \log_2 7 = 2.807$ bits / message symbol

$q=8$ $\quad H(S)_{max} = \log_2 8 = 3$ bits / message symbol

$q=9$ $\quad H(S)_{max} = \log_2 9 = 3.17$ bits / message symbol

$q=10$ $\quad H(S)_{max} = \log_2 10 = 3.322$ bits / message symbol

Properties of Entropy

Continued

4. Property of Additivity

Suppose that we split symbols S_q into 'n' sub symbols such that $S_q = S_{q_1}, S_{q_2}, \ldots, S_{q_n}$ occurring with probabilities $P_{q_1}, P_{q_2} \ldots P_{q_n}$ such that $P_q = P_{q_1} + P_{q_2} + \ldots + P_{q_n}$

$$= \sum_{i=1}^{n} P_{q_i}$$

then the probability will not be reduced.

Derivation - Assignment

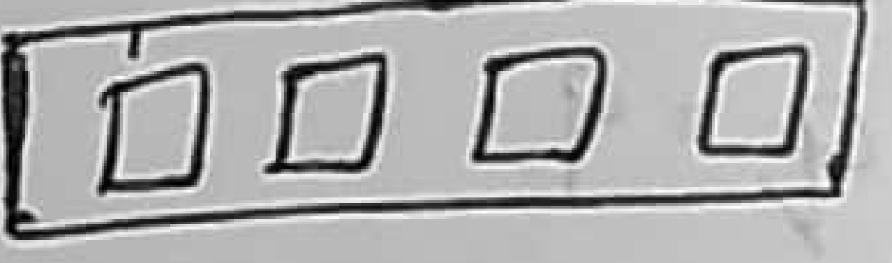
5. Source efficiency and redundancy

$$\eta_s = \frac{H(S)}{H(S)_{max}}$$

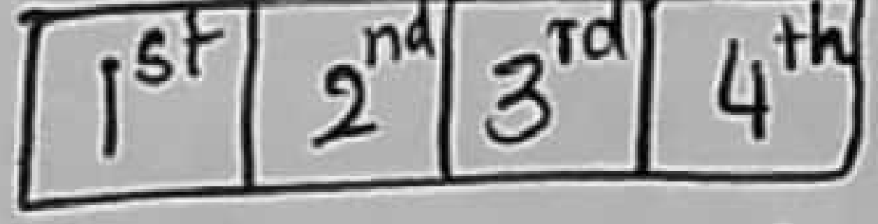
Counter part of source efficiency is source redundancy $R_{\eta_s} = 1 - \eta_s$

Problem

① A certain data source has 8 symbols that are produced in blocks of 4 at a rate of 500 blocks/s. The 1st symbol in each block is always the same. The remaining 3 are filled by any of the 8 symbols with equal probability. What is the entropy rate of this source?



Sol:- Probability of 1st symbol [i.e is same in each block] = 1 ∴ Information = 0.



$H_1 = 0$

$P_1 = 1$

$P_2 = \frac{1}{8}$ × $\Big[$Entropy $= \sum_{i=1}^{4} P_i \log \frac{1}{P_i}$

$P_3 = \frac{1}{8}$

$P_4 = \frac{1}{8}$

$= 1 \log 1 + \frac{1}{8} \log 8 + \frac{1}{8} \log 8 + \frac{1}{8} \log 8$

Since P_2, P_3, P_4 have equal

$= \frac{9}{8} \Big]$ × probability, they are emitting the max. information

q = 8 = no. of symbols

$H(2)_{max} = \log_2 q = \log_2 8 = 3$ bits/ Msymbols ~~Hblock~~

$H(3)_{max} = \log_2 q = \log_2 8 = 3$ bits / ~~MS~~ block

$H(4)_{max} = \log_2 q = \log_2 8 = 3$ bits / ~~MS~~ block

$H = H(2)_{max} + H(3)_{max} + H(4)_{max} + H(1)$

$H = 3 + 3 + 3 + 0 = 9$ bits / ~~MS~~ block

$R_s = r_s \times H(S)$

$= 500$ ~~bits~~ blocks/s × 9 bits / block

$R_s = 4500$ bits/sec

② A black & white TV picture consists of 525 lines of picture information. Assume that each line consists 525 picture elements (Pixel) and that each element can have 256 brightness levels. Pictures are repeated at the rate of 30 frames/second. Calculate the average rate of information conveyed by a TV set.

Sol :- Aspect ratio $\frac{W}{L} = 4:3$ If width $= 40$ inch then length $= 30$ inches

Total no. of pixels in 1 frame $= 525 \times 525$

$= 275625$ pixels

Total no. of different frames possible

$= (256)^{275625}$ frames

All the frames are occurring with equal probability

$$H(S)_{max} = \log_2 q = \log_2 (256)^{275625}$$

$$= 275625 \log_2 256$$

$$H(S)_{max} = 2205000 \text{ bits/frame}$$

$$R_s = r_s H(S)$$

$$= 30 \text{ frames/s} \times 2205000 \text{ bits/frame}$$

$$R_s = 66.15 \times 10^6 \text{ bits/sec}$$

③ Find the information content of a message that consists of a digital word of 9 digits long in which each digit may take 1 of 5 possible levels. The probability of sending any of the 5 levels is assumed to be equal and the level in any digit does not depend on the values taken by previous digits.

6/8

Sol^n length of digital word = 9 digits

Number of levels = 5 equiprobable.

$$H_T = H_1 + H_2 + --- + H_9$$

$$H = H(S)_{max} = \log_2 q = \log_2 5 = 2.322 \text{ bits / level}$$

$$H_1 = H_2 = --- H_9 = 2.322 \text{ bits / level}$$

∴ Total information content

$$H_T = 9(2.322)$$

$$H_T = 20.898 \text{ bits / level}$$

Property of additivity

Statement continued

Then, the splitted symbol entropy is

$$H'(S) = H(P_1, P_2 --- P_{q-1}, P_{q_1}, P_{q_2} --- P_{q_n})$$

$$= \sum_{i=1}^{q-1} P_i \log \frac{1}{P_i} + \sum_{j=1}^{n} P_{q_j} \log \frac{1}{P_{q_j}}$$

$$= \sum_{i=1}^{q} P_i \log \frac{1}{P_i} - P_q \log \frac{1}{P_q} + \sum_{j=1}^{n} P_{q_j} \log \frac{1}{P_{q_j}}$$

WKT $\sum_{j=1}^{n} P_{q_j} = 1$ & $P_q . P_{q_j} = P_{q_j}$

$$H'(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} - \sum_{j=1}^{n} P_{q_j} \log \frac{1}{P_{q_j}} + \sum_{j=1}^{n} P_{q_j} \log \frac{1}{P_{q_j}}$$

$$= \sum_{i=1}^{q} P_i \log \frac{1}{P_i} + P_q \sum_{j=1}^{n} \frac{P_{aj}}{P_q}\left[\log \frac{P_a}{P_{q_j}}\right]$$

$\therefore H'(S) = H(S) +$ a positive quantity since $P_{qj} \leq P_q$ for all j

$$\therefore H'(S) \geq H(S)$$

Partitioning of symbols into sub-symbols cannot decrease the entropy.

④ Suppose the symbols S_1 and S_2 are 2 zero memory sources with probabilities $P_1, P_2 \cdots P_n$ for S_1 and $q_1, q_2 \cdots q_n$ for S_2. Show that the entropy of source S_1 $H(S_1) \leq \sum_{k=1}^{n} P_k \log\left(\frac{1}{q_k}\right)$

<u>Sol</u>

Sol^n :- $\quad H(S_1) = \sum_{k=1}^{n} P_k \log \frac{1}{P_k}$ — ①

WKT

$$\sum_{k=1}^{n} P_k = 1 \quad — ②$$

$$H(S_2) = \sum_{k=1}^{n} q_k \log \frac{1}{q_k} \quad — ③$$

$$\sum_{k=1}^{n} q_k = 1 \quad — ④$$

Let us consider $H(S_1) - \sum_{k=1}^{n} P_k \log \frac{1}{q_k}$

$$= \sum_{k=1}^{n} P_k \log \frac{1}{P_k} - \sum_{k=1}^{n} P_k \log \frac{1}{q_k}$$

$$= \sum_{k=1}^{n} P_k \left(\log \frac{1}{P_k} - \log \frac{1}{q_k}\right)$$

$$= \sum_{k=1}^{n} P_k \log \frac{q_k}{P_k}$$

$$= \sum_{k=1}^{n} P_k \frac{\ln q_k / P_k}{\ln 2}$$

9/8

$$= \log_2 e \sum_{k=1}^{n} P_k \ln \frac{q_k}{P_k} \; - ⑤$$

From logarithmic & straight line equation

~~$\ln\left(\frac{1}{x}\right) \geq 1-x$~~

$$\ln x \leq x-1$$

let $x = \frac{q_k}{P_k}$

$$\ln \frac{q_k}{P_k} \leq \left[\frac{q_k}{P_k}-1\right]$$

X^{ly} both sides by P_k taking $\Sigma\, k=1$ for all 1 to n then multiplying by $\log_2 e$

$$\log_2 e \sum_{k=1}^{n} P_k \ln \frac{q_k}{P_k} \leq \log_2 e \sum_{k=1}^{n} P_k \left(\frac{q_k}{P_k}-1\right) \; - ⑥$$

~~$\left(\frac{q_k}{P_k}-1\right)$~~ $-$ ⑥

Comparing eq ⑥ and eq ⑤

RHS of eq ⑤ = LHS of eq ⑥

$$H(S) - \sum_{k=1}^{n} P_k \log \frac{1}{q_k} \leq \log_2 e \left(\sum_{k=1}^{n} q_k - \sum_{k=1}^{n} P_k\right)$$

$$H(S) - \sum_{k=1}^{n} P_k \log \frac{1}{q_k} \leq 0$$

$$H(S) \leq \sum_{k=1}^{n} P_k \log \frac{1}{q_k}$$ Hence ~~shor~~ proved.

⑤ A pair of dice are tossed simultaneously. The outcome of I dice is recorded as x_1 and that of the II dice as x_2. 2 events are defined as follows

$A = \{(x_1, x_2)$ such that $(x_1 + x_2) \leq 7\}$

$B = \{(x_1, x_2)$ such that $x_1 > x_2\}$

9/8

Which event conveys more Information? Support your answers by numerical computation.

Sol^n :- $S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$

$(2,1)\ -\ -\ -\ -\ -\ -\ -\ (2,6)$

$(3,1)\ -\ -\ -\ -\ -\ -\ -\ (3,6)$

$(4,1)\ -\ -\ -\ -\ -\ -\ -\ (4,6)$

$(5,1)\ -\ -\ -\ -\ -\ -\ -\ (5,6)$

$(6,1)\ -\ -\ -\ -\ -\ -\ -\ (6,6)\}$

$x_1 = \frac{21}{36}$ outcome of I dice $\quad x_2$ = outcome of II dice

$A = \{(1,1)(1,2)(1,3)(1,4)(1,5)(1,6)$

$(2,1)(2,2)(2,3)(2,4)(2,5)$

$(3,1)(3,2)(3,3)(3,4)$

$(4,1)(4,2)(4,3)$

$(5,1)(5,2)(6,1)\}$

$A = \frac{21}{36}$

$B = \{(2,1)(3,1)(3,2)(4,1)(4,2)(4,3)(5,1)(5,2)(5,3)(5,4)$

$(6,1)(6,2)(6,3)(6,4)(6,5)\}$

$B = \frac{15}{36}$

$I_1 = \log_2 \frac{1}{\frac{21}{36}} = 0.77$ bits/ms

$I_2 = \log_2 \frac{1}{15/36} = 1.266$ bits/ms

⑥ A discrete source S emits 2 independent Images I_1, I_2 with probabilities 0.55, 0.45 respectively. Calculate the efficiency of the source and its redundancy.

Sol :- $H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i}$

$= P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}$

$H(S)_{max} = \log_2 q$

$= \log_2 2$

$H(S)_{max} = 1$ bit/ms

9/8

$$= 0.55 \log \frac{1}{0.55} + 0.45 \log \frac{1}{0.45}$$

$$= 0.4743 + 0.5184$$

$H(S) = 0.9927$ bits / images

$$\eta_s = \frac{H(S)}{H(S)_{max}} = \frac{0.9927}{1} \times 100 = 99.27\%$$

Redundancy $\rightarrow$ $R_{\eta_s} = 1 - \eta_s$

$$= 1 - 99.27\%$$

$$= 0.73\%$$

Extension of zero memory source

$H(S) = \{S_1, S_2\}$

$P(S) = \{P_1, P_2\}$

Second extension

$(S_1 S_2)$ $(P_1 P_2)$

$S_1 S_1$ occurring with probability $P_1 P_1$

$S_1 S_2 \rightarrow P_1 P_2$

$S_2 S_1 \rightarrow P_2 P_1$

$S_2 S_2 \rightarrow P_2 P_2$

$$H(S) = \sum_{K=1}^{2} P_K \log \frac{1}{P_K} = P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}$$

Second extension is represented by $H(S^2)$

$$H(S^2) = P_1^2 \log\left(\frac{1}{P_1^2}\right) + P_1 P_2 \log \frac{1}{P_1 P_2} + P_2 P_1 \log \frac{1}{P_2 P_1} + P_2^2 \log\left(\frac{1}{P_2^2}\right)$$

$$= 2P_1^2 \log\left(\frac{1}{P_1}\right) + 2 P_1 P_2 \log \frac{1}{P_1 P_2} + 2P_2^2 \log \frac{1}{P_2}$$

$$= 2\left(P_1^2 \log \frac{1}{P_1} + P_1 P_2 \log \frac{1}{P_1 P_2} + P_2^2 \log \frac{1}{P_2}\right)$$

9|8

$H(S^2) = 2\left[P_1 (P_1+P_2) \log \frac{1}{P_1} + P_2 (P_1+P_2) \log \frac{1}{P_2} \right]$

let $P_1 + P_2 = 1$

$$H(S^2) = 2\left(P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}\right)$$

$$H(S^2) = 2H(S)$$

IIIly for 3rd extension source

$$H(S^3) = 3H(S)$$

$$H(S^4) = 4H(S)$$

$$\vdots$$

$$\boxed{H(S^n) = nH(S)}$$

Q. Compute the 2nd extension source and prove that it is twice of H(S) for the previous problem.

Soln:- $H(S^2) = P_1^2 \log \frac{1}{P_1^2} + 2P_1P_2 \log \frac{1}{P_1P_2} + P_2^2 \log \frac{1}{P_2^2}$

$$= 2 \times .3025 \log \frac{1}{0.3025} + 2\times 0.2475 \log \frac{1}{0.2475}$$

$$+ 0.2025 \log \frac{1}{0.2025}$$

$$= 0.5218 + 0.99717 + 0.4667$$

$$H(S^2) = 1.98553$$

$$2H(S) = 2 \times 0.9927$$

$$2H(S) = 1.98553$$

Hence proved $H(S^2) = 2H(S)$

Problems

① Consider a zero memory source emitting 3 symbols x, y and z with respective probabilities 0.6, 0.3, 0.1

i) Calculate entropy of source

ii) Find all the symbols and corresponding probabilities of 2nd order extension source.

iii) 2^n Entropy of 2^nd order extension source 13/8

iv) Show that $H(S^2) = 2H(S)$

Sol^n :- i) Entropy $= \sum_{i=1}^{3} P_i \log \frac{1}{P_i}$

$= P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2} + P_3 \log \frac{1}{P_3}$

$= 0.6 \log \frac{1}{0.6} + 0.3 \log \frac{1}{0.3} + 0.1 \log \frac{1}{0.1}$

= 0.~~07~~442 + 0.521 + 0.332

= 1.296 bits / symbol

ii) q^n $3^2 = 9$ symbols

$H(S^2) =$ \ ~~$P_1 \log$~~

$\underbrace{(x \cdot y\ z)}$
$\uparrow$
given symbols

∵ it is 2^nd extension multiply by (x y z)

$\overset{0.6}{x}\ \overset{0.3}{y}\ \overset{0.1}{z}$ $(x\ y\ z)(x\ y\ z) = \{x^2, xy, xz, yx, y^2, yz, zx, zy, z^2\}$

(probabilities above: 0.6 0.3 0.1 0.6 0.3 0.1)

$P[xx] = 0.6 \times 0.6 = 0.36$

$P[xy] = 0.6 \times 0.3 = 0.18$

$P[xz] = 0.6 \times 0.1 = 0.06$

$P[yx] = 0.3 \times 0.6 = 0.18$

$P[yy] = 0.3 \times 0.3 = 0.09$

$P[yz] = 0.3 \times 0.1 = 0.03$

$P[zx] = 0.1 \times 0.6 = 0.06$

$P[zy] = 0.1 \times 0.3 = 0.03$

$P[zz] = 0.1 \times 0.1 = 0.01$

iii) $H(S^2) = \sum_{i=1}^{9} P_i \log \frac{1}{P_i}$

$= 0.36 \log \frac{1}{0.36} + 0.18 \log \frac{1}{0.18} + 0.06 \log \frac{1}{0.06} + 0.18 \log \frac{1}{0.18} +$

$0.09 \log \frac{1}{0.09} + 0.03 \log \frac{1}{0.03} + 0.06 \log \frac{1}{0.06} + 0.03 \log \frac{1}{0.03}$

13/8.

$+ 0.01 \log \frac{1}{0.01}$

$= 2.591$ bits /MS

iv) $H(S^2) = 2H(S)$

$= 2(1.296)$

$= 2.592$

② A source emits 4 symbols M_1, M_2, M_3, M_4 with probabilities $(\frac{7}{16}, \frac{5}{16}, \frac{1}{8}, \frac{1}{8})$ respectively. Find

i) entropy of the source

ii) list all the 2^{nd} order symbol and corresponding probabilities

iii) Find entropy of 2^{nd} order extension source

iv) Prove that $H(S^2) = 2H(S)$

Solⁿ:- i) $H(S) = \sum_{i=1}^{4} P_i \log \frac{1}{P_i}$

$= \frac{7}{16} \log \frac{16}{7} + \frac{5}{16} \log \frac{16}{5} + \frac{1}{8} \log 8 + \frac{1}{8} \log 8$

$H(S) = 1.7962$ bits / MS.

ii) $q^n \quad 4^2 = 16$ symbols

$(M_1 M_2 M_3 M_4)(M_1 M_2 M_3 M_4)$

$= [M_1M_1, M_1M_2, M_1M_3, M_1M_4, M_2M_1, \ldots\ldots M_4M_3, M_4M_4)$

$P[M_1M_1] = \frac{7}{16}^2 = \frac{49}{256}$

$P[M_1M_2] = \frac{7}{16} \times \frac{5}{16} = \frac{35}{256}$

$P[M_1M_3] = \frac{7}{128}$

$P[M_1M_4] = \frac{7}{128}$

$P[M_2M_1] = \frac{35}{256}$

$P[M_2M_2] = \frac{25}{256}$

$P[M_2M_3] = \frac{5}{128}$

$P[M_2M_4] = \frac{5}{128}$

13/8

$P[M_3M_1] = \frac{7}{128}$ $\qquad$ $P[M_4M_1] = \frac{7}{128}$

$P[M_3M_2] = \frac{5}{128}$ $\qquad$ $P[M_4M_2] = \frac{5}{128}$

$P[M_3M_3] = \frac{1}{64}$ $\qquad$ $P[M_4M_3] = \frac{1}{64}$

$P[M_3M_4] = \frac{1}{64}$ $\qquad$ $P[M_4M_4] = \frac{1}{64}$

iii) $H(S^2) = \sum_{i=1}^{16} P_i \log \frac{1}{P_i}$

$= (\overset{\log_{10}}{\cancel{1.08149 \times 3.32219}}$ $\qquad$ $\overset{\log_{10}}{(1.0814)} \div \log_{10} 2$

$H(S^2) = 3.5923$ bits/MS

iv) $H(S^2) = 2H(S)$

$= 2 \times 1.7962$

$3.5923 = 3.5924$

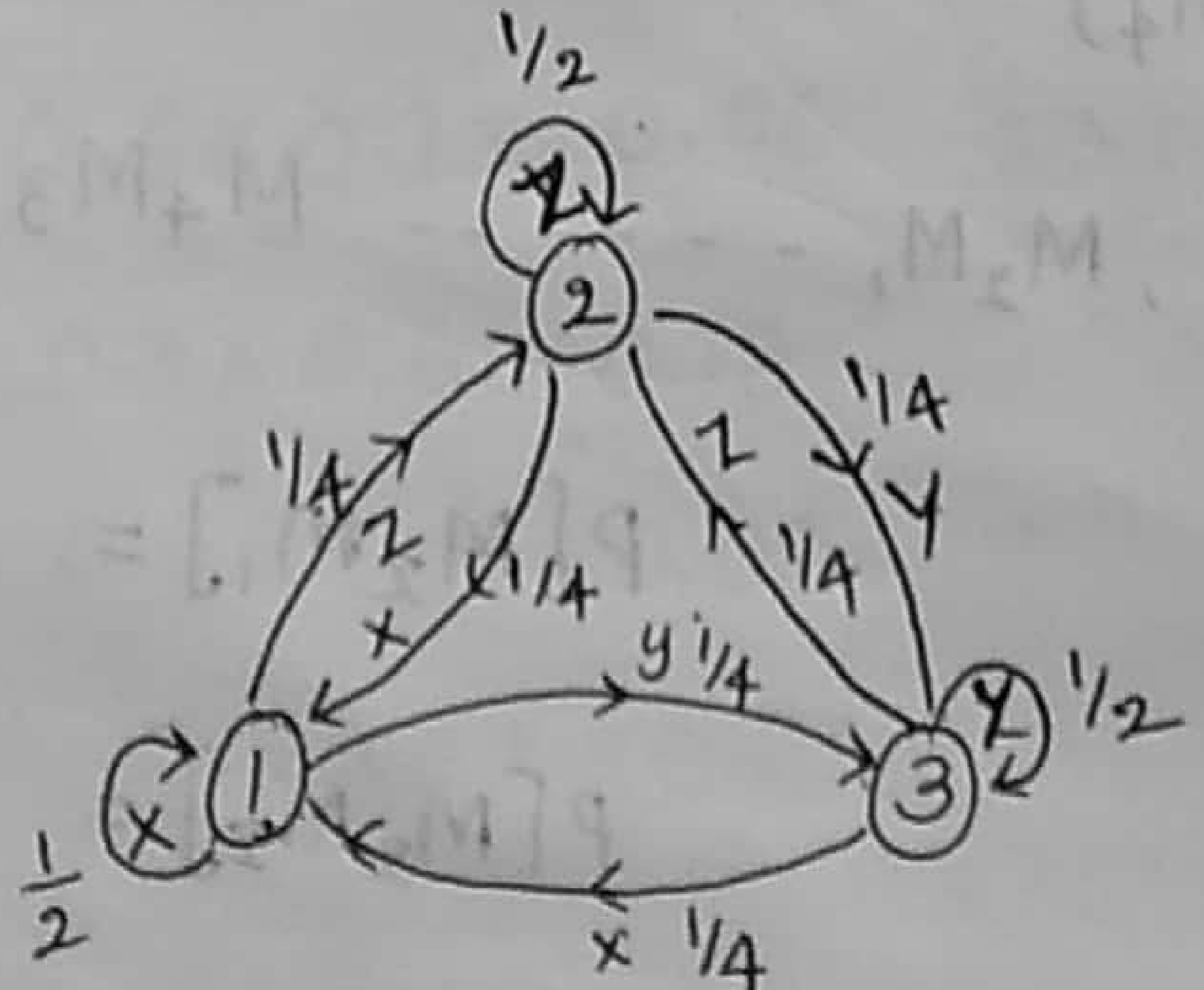
③ For the same problem find 3rd order extension entropy

<u>Sol^n</u> $H(S^2) = 3H(S) = 3 \times 1.7962 = 5.3886$ bits/MS

<u>Dependent Sources</u> or **<u>Memory source</u>**

<u>Markoff model</u>

It is represented using either state diagram or free diagram.



$P(1) = \frac{1}{3}$

$P(2) = \frac{1}{3}$

$P(3) = \frac{1}{3}$

$P(x) = P(1)\,P(1\to1) \underset{\text{or}}{\underset{\downarrow}{+}} P(2)\,P(2\to1) + P(3)\,P(3\to1)$

$= \frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{4} = \frac{1}{6}+\frac{1}{12}+\frac{1}{12} = \frac{1}{3}$

13/8

2nd order Markov source

$P(XX) = P(1\to1\to1)$ or $P(2\to1\to1)$ or $\cdot$ $P(3\to1\to1)$

$= P(1\to1\to1) + P(2\to1\to1) + P(3\to1\to1)$

$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$

$= \frac{1}{12} + \frac{1}{24} + \frac{1}{24}$

$P(XX) = \frac{1}{6}$

$P(XY) = P(1\to1\to3) + P(3\to3\to3) + P(2\to1\to3)$

$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{24} + \frac{1}{48} + \frac{1}{48} = \frac{1}{16} + \frac{1}{48} = \frac{1}{12}$

$P(YX) = P(1\to3\to1) + P(3\to3\to1) + P(2\to3\to1)$

$= \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$

$P(XZ) = P(3\to1\to2) + P(2\to1\to2) + P(1\to1\to2)$

$= \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + P\left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$

$P(YY) = P(3\to3\to3) + P(1\to3\to3) + P(2\to3\to3)$

$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$

$P(YZ) = P(2\to3\to2) + P(3\to3\to2) + P(1\to3\to2)$

$= \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$

$P(ZX) = P(3\to2\to1) + P(2\to2\to1) + P(1\to2\to1)$

$= \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + P\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \frac{1}{12}$

$P(ZY) = P(2\to2\to3) + P(3\to2\to3) + P(1\to2\to3)$

$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$

16/8

$P(ZZ) = P(2 \to 2 \to 2) + P(1 \to 2 \to 2) + P(3 \to 2 \to 2)$

$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{6}$

Entropy and Information rate of Mark off sources

Hi indicates entropy of i^{th} state

$$H_i = \sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}} \text{ bits/MS}$$

$$H = \sum_{i=1}^{n} P_i H_i = \sum_{i=1}^{n} P_i \left[\sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}} \right] \text{ bits/MS}$$

H → entropy of the source.

Information rate $R_s = r_s H$ bits/sec

r_s → number of state transitions per second

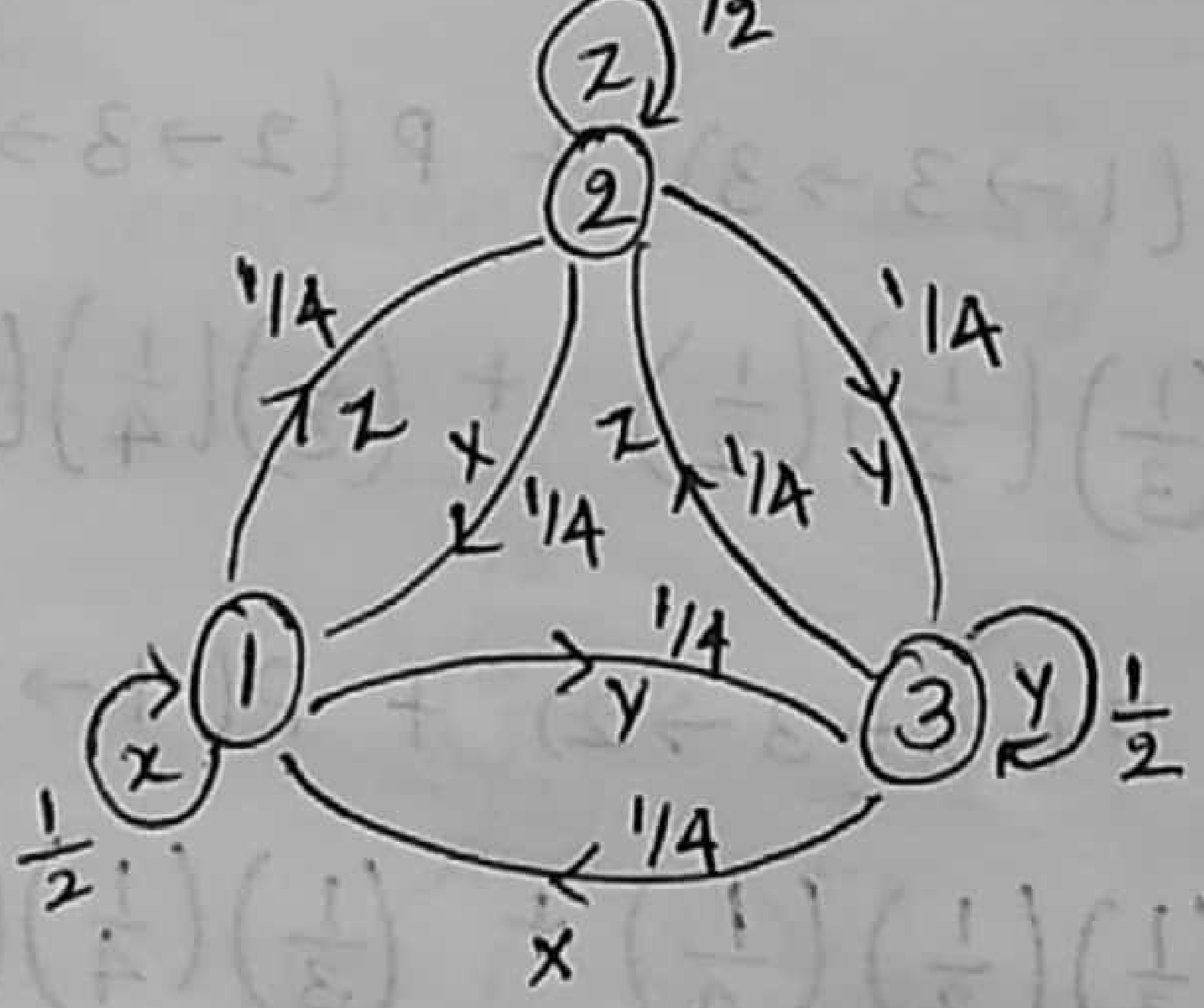
<u>or</u> symbol rate of the source

Problem

① For the mark off source of figure below find

i) entropy of each state

ii) entropy of the source



Given : $P(1) = P(2) = P(3) = \frac{1}{3}$

Sol :- 3 states ; Find the state entropy first

$$H_i = \sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}} \qquad i=1 ; j = 1, 2, 3$$

$$H_1 = P_{11} \log \frac{1}{P_{11}} + P_{12} \log \frac{1}{P_{12}} + P_{13} \log \frac{1}{P_{13}}$$

$$H_1 = \frac{1}{2} \log 2 + \frac{1}{4} \log 4 + \frac{1}{4} \log 4 = \frac{1}{2} + \frac{2}{4} + \frac{2}{4}$$

$= \frac{3}{2}$ bits / MS 16/8

$$H_2 = P_{21} \log \frac{1}{P_{21}} + P_{22} \log \frac{1}{P_{22}} + P_{23} \log \frac{1}{P_{23}}$$

$$= \frac{1}{4} \log 4 + \frac{1}{2} \log 2 + \frac{1}{4} \log 4 = \frac{3}{2} \text{ bits / MS}$$

$$H_3 = P_{31} \log \frac{1}{P_{31}} + P_{32} \log \frac{1}{P_{32}} + P_{33} \log \frac{1}{P_{33}}$$

$$= \frac{1}{4} \log 4 + \frac{1}{4} \log 4 + \frac{1}{2} \log 2 = \frac{3}{2} \text{ bits / MS}$$

$$H = \sum_{i=1}^{n} P_i H_i = \overset{\checkmark}{\times} \left[\sum_{i=1}^{n} P_i \left[\sum_{j=1}^{n} P_{ij} \log \frac{1}{P_{ij}} \right] \right.$$

$$= P_1 H_1 + P_2 H_2 + P_3 H_3$$

$$= \left(\frac{1}{3}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{3}\right)\frac{3}{2}$$

$$= 1.5 \text{ bits / MS} \left.\right] \times \checkmark$$

$$\times \left[H = H_1 + H_2 + H_3 = 4.5 \text{ bits / MS} \right] \times$$

19/8

Theorem :- If $P(m_i)$ is the probability of a sequence m_i of N symbols from the source and if $G_N = \frac{1}{N} \sum P(m_i)$

$\log \frac{1}{P_{m_i}} = \frac{1}{N} H(\bar{S}^N)$ $\quad G_N = \frac{1}{N} \sum_{i=1}^{N} P(m_i) \log \frac{1}{P_{m_i}} = \frac{1}{N} H(\bar{S}^N)$

where, $H(\bar{S}) \rightarrow$ is the entropy of adjoint source.

$G_N \rightarrow$ monotonically decreasing function of N

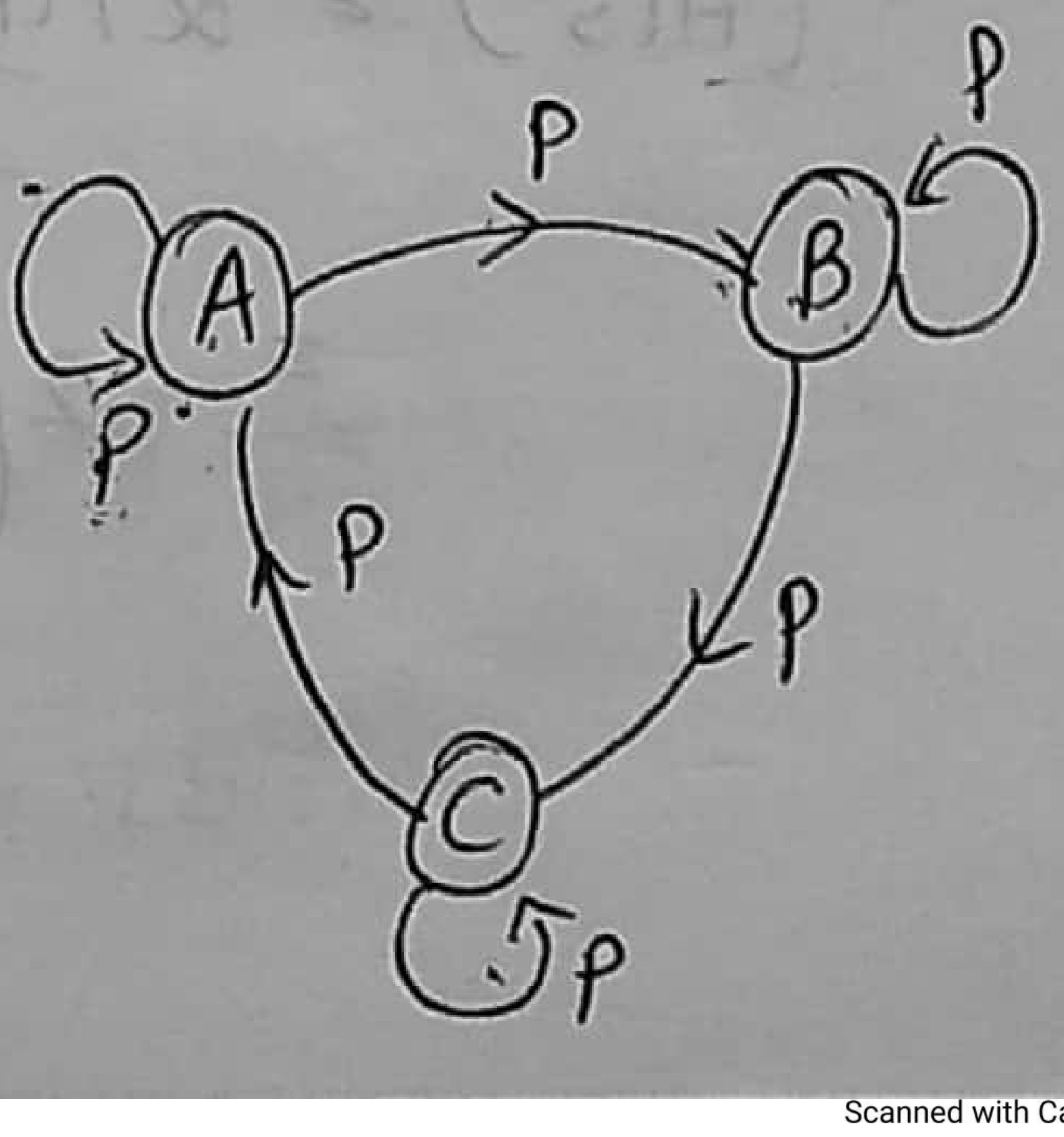
$$\lim_{N \to \infty} G_N = H \text{ bits / sec}$$

Problem :-

1) For the first order Markov source with first order source $S = \{A, B, C\}$ shown in figure below.

i) Compute the probabilities of state.

ii) Find $H(S)$ and $H(S^2)$



Sol^n :-

i) $P(A) = pP(A) + pP(C)$ — ①

$P(B) = pP(B) + pP(A)$ — ②

$P(C) = pP(C) + pP(B)$ — ③

adding equations ①, ② & ③

$$P(A) + P(B) + P(C) = 2pP(A) + 2pP(B) + 2pP(C)$$

$$P(A) + P(B) + P(C) = 2p[P(A) + P(B) + P(C)]$$

$$\boxed{p = \frac{1}{2}}$$ — ④

① ⇒ $P(A) = \frac{1}{2}P(A) + \frac{1}{2}P(C)$

$$\frac{1}{2}P(A) = \frac{1}{2}P(C)$$

$P(A) = P(C)$ — ⑤

$$P(B) = \frac{1}{2}P(B) + \frac{1}{2}P(A)$$

$$\frac{1}{2}P(B) = \frac{1}{2}P(A)$$

$P(B) = P(A)$ — ⑥

$$\therefore P(A) = P(B) = P(C) = \frac{1}{3}$$

$$H_i = \sum_{j=A}^{C} P_{ij} \log \frac{1}{P_{ij}}$$

$i=A$, $H_A = P_{AA} \log \frac{1}{P_{AA}} + P_{AB} \log \frac{1}{P_{AB}} + P_{AC} \log \frac{1}{P_{AC}}$

$= \frac{1}{2} \log 2 + \frac{1}{2} \log 2 + 0$

$H_A = 1$ bit/MS

$i=B$, $H_B = P_{BA} \log \frac{1}{P_{BA}} + P_{BB} \log \frac{1}{P_{BB}} + P_{BC} \log \frac{1}{P_{BC}}$

$= 0 + \frac{1}{2} \log 2 + \frac{1}{2} \log 2$

$H_B = 1$ bit/MS

$i=C$, $H_C = P_{CA} \log \frac{1}{P_{CA}} + P_{CB} \log \frac{1}{P_{CB}} + P_{CC} \log \frac{1}{P_{CC}}$

$H_C = \frac{1}{2} \log 2 + 0 + \frac{1}{2} \log 2 = 1$ bit/MS

19/8

$H = P_A H_A + P_B H_B + H_c P_c = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3}$ bits/MS

$= 1$ bit/MS

$H(S^2) = 2H(S)$

$= 2 \times 1 = 2$ bits/MS

② Find $G(1)$, $G(2)$, $G(3)$ for the previous problem
G_1 G_2 G_3

Sol^n :-

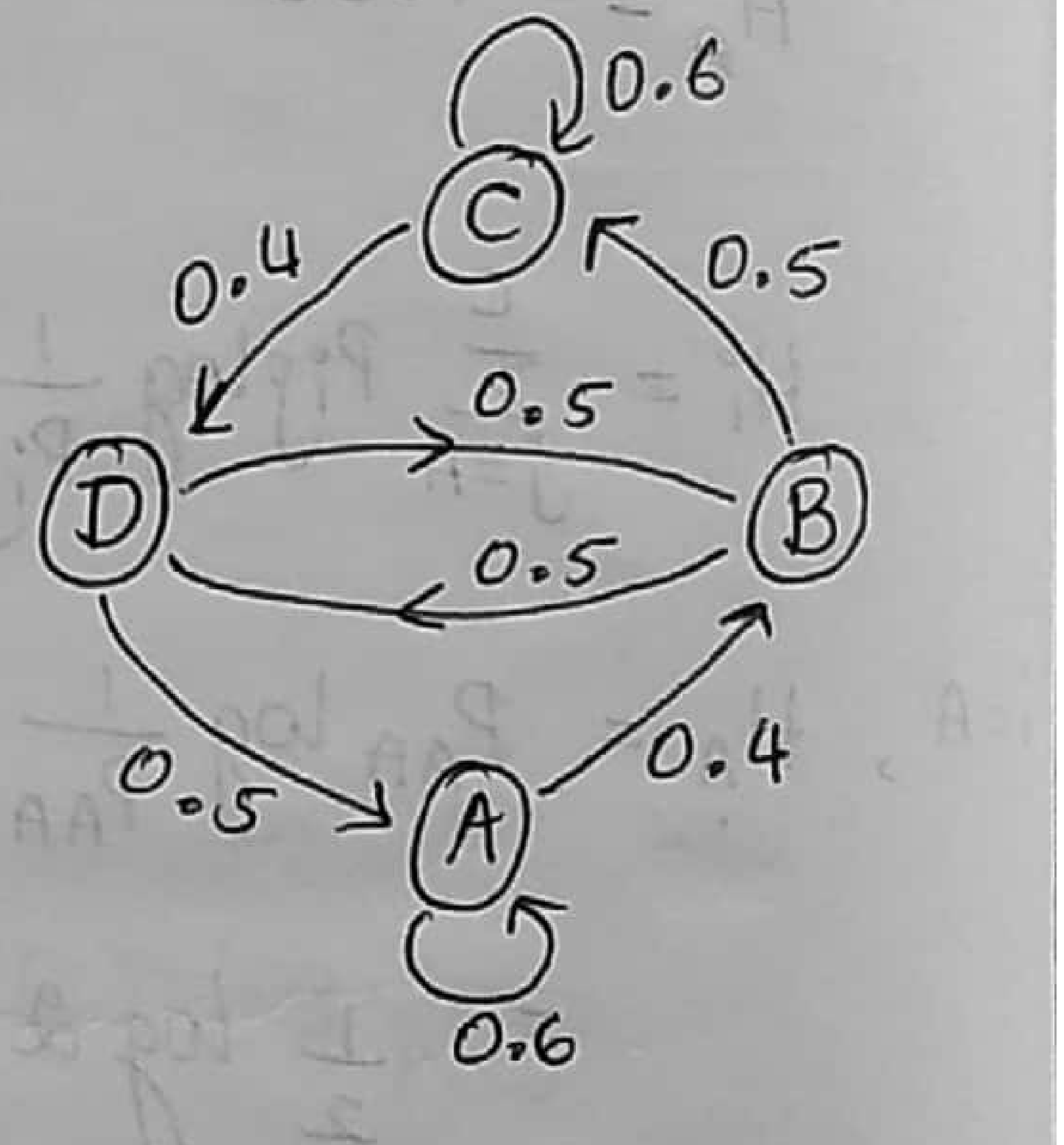
$$G_N = \frac{1}{N} \sum_i P_{(mi)} \log \frac{1}{P_{(mi)}}$$

~~Ex~~ $G_N = \frac{1}{N} H(\bar{S}^N)$

$N=1$ $G_1 = \frac{1}{1} H(\bar{S}^1) = 1$ bit/MS

$G_2 = \frac{1}{2} H(\bar{S}^2) = \frac{1}{2} \times 2 = 1$ bit/MS

③ Consider the state diagram of Markov source of figu
below. i) Compute the state probabilities
ii) Find the entropy of each state
iii) Find the entropy of source



Sol^n :- 4 states → A, B, C, D

i) $P(A) = 0.6\,P(A) + 0.5\,P(D)$ — ①

$P(B) = 0.4\,P(A) + 0.5\,P(D)$ — ②

$P(C) = 0.6\,P(C) + 0.5\,P(B)$ — ③

$P(D) = 0.4\,P(C) + 0.5\,P(B)$ — ④

Adding ① & ②

$P(A) + P(B) = 1.0\,P(A) + 1.0\,P(D)$ — ⑤

③ + ④ $P(C) + P(D) = 1.0\,P(C) + 1.0\,P(B)$ — ⑥

~~P(A)~~ ① ⇒ $+0.4\,P(A) = 0.5\,P(D)$

$P(A) = \frac{+0.5}{0.4} P(D)$

19/8

③ $\Rightarrow +0.4P(C) = 0.5P(B)$

$$P(C) = \frac{+0.5}{0.4}P(B)$$

$$P(D) = \frac{+0.5}{0.4}P(B) \times 0.4 + 0.5P(B)$$

$$P(D) = 01\,P(B)$$

$$P_A + P_B + P_C + P_D = 1 \qquad P_A = \frac{0.5}{0.4}P_B$$

$$P_A + P_B + P_C + P_B = 1 \qquad P_C = \frac{0.5}{0.4}P_B$$

$$P_A + 2P_B + P_C = 1$$

$$\frac{0.5}{0.4}P_B + 2P_B + \frac{0.5}{0.4}P_B = 1$$

$$4.5P_B = 1$$

$$P_B = \frac{1}{4.5} = \frac{2}{9}$$

$$P_A = \frac{0.5}{0.4}\left(\frac{2}{9}\right) = \frac{5}{18}$$

$$P_C = \frac{0.5}{0.4}\left(\frac{2}{9}\right) = \frac{5}{18}$$

$$P_B = P_D = \frac{2}{9}$$

ii) Entropies of state

$$H_i = \sum_{j=A}^{D} P_{ij} \log \frac{1}{P_{ij}}$$

$i = A$ $$H_A = P_{AA}\log\frac{1}{P_{AA}} + P_{AB}\log\frac{1}{P_{AB}} + P_{AC}\log\frac{1}{P_{AC}} + P_{AD}\log\frac{1}{P_{AD}}$$

$$= 0.6\log\frac{1}{0.6} + 0.4\log\frac{1}{0.4} + 0 + 0$$

$H_A = 0.971$ bits / M.S

$i = B$ $$H_B = P_{BA}\log\frac{1}{P_{BA}} + P_{BB}\log\frac{1}{P_{BB}} + P_{BC}\log\frac{1}{P_{BC}} + P_{BD}\log\frac{1}{P_{BD}}$$

$$= 0 + 0 + 0.5\log\frac{1}{0.5} + 0.5\log\frac{1}{0.5}$$

$H_B = 1$ bit / MS

19|8

$i = C \quad H_C = P_{CA} \log \frac{1}{P_{CA}} + P_{CB} \log \frac{1}{P_{CB}} + P_{CC} \log \frac{1}{P_{CC}} + P_{CD} \log \frac{1}{P_{CD}}$

$= 0 + 0 + 0.6 \log \frac{1}{0.6} + 0.4 \log \frac{1}{0.4}$

$H_C = 0.971$ bits/MS

$i = D \quad H_D = 0.5 \log \frac{1}{0.5} + 0.5 \log \frac{1}{0.5} + 0 + 0$

$H_D = 1$ bit/MS

iii) Entropy of the source

$H = \sum P_i H_i = P_A H_A + P_B H_B + P_C H_C + P_D H_D$

$= \frac{5}{18}(0.971) + \frac{2}{9}(1) + \frac{5}{18}(0.971) + \frac{2}{9}(1)$

$H = 0.9838$ bits/MS.

2018

④ You are asked to design an information system which gives the information every year for about 200 students passing out BE ECE degree from a certain university. The students can get into one of the 3 fields as given below.

i) Go abroad for higher studies. – Ⓐ

ii) Join MBA or civil services – Ⓑ

iii) Join industries in India – Ⓒ

Based on the data given below construct the model for the source and source entropy.

a) On an average 100 students are going abroad.

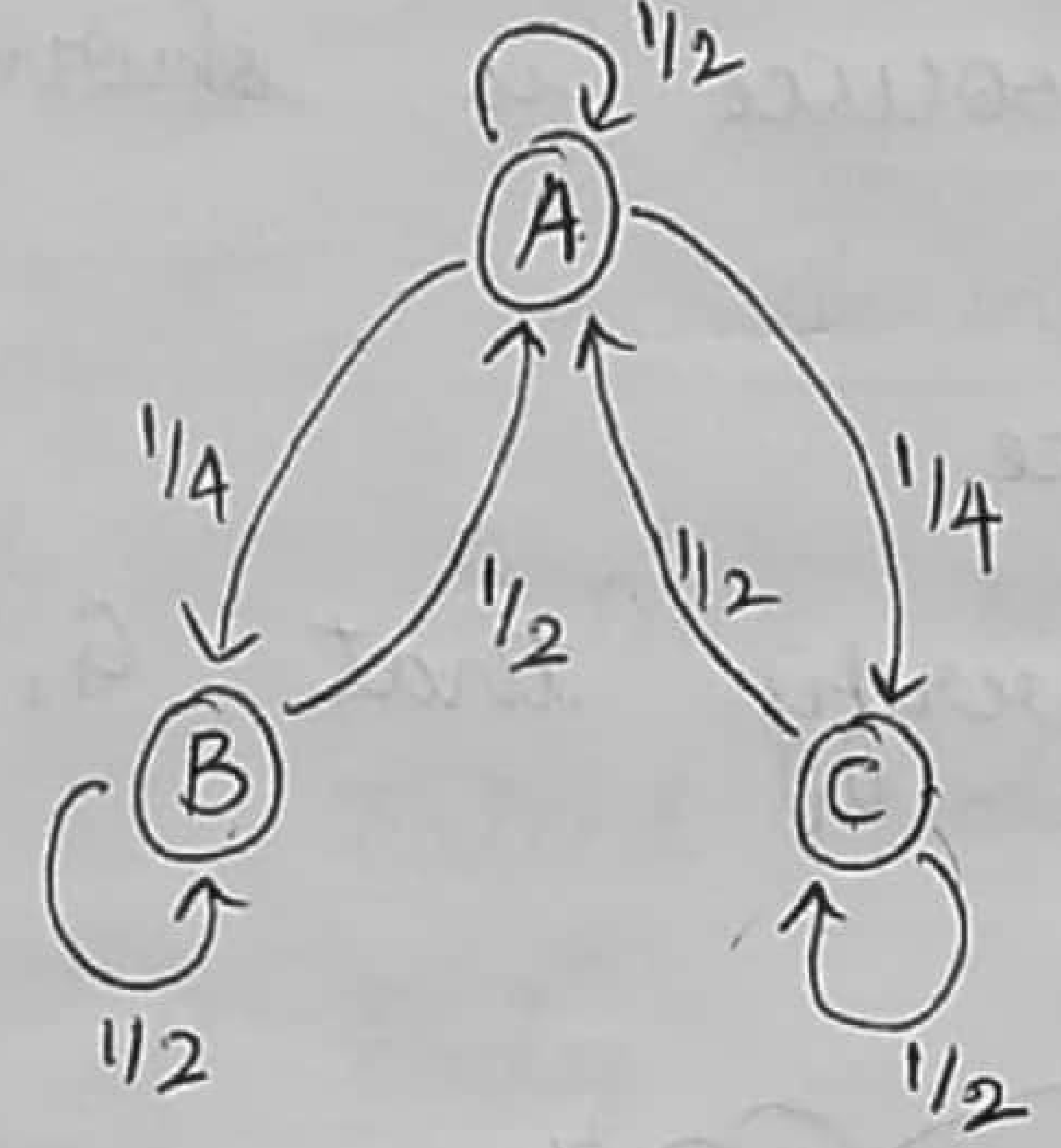
b) Out of 100 going abroad this year, 50 were reported going abroad next year, while 25 each went to MBA and civil services or joined industries in India.

c) Out of 100 remaining in India this year 50 continue to do so, while 50 went abroad next year.

d) Those joining MBA and civil services or industry couldn't swap the 2 fields next year.

2018

Sol^n:-



$$P(A) = \frac{1}{2} P(A) + \frac{1}{2} P(B) + \frac{1}{2} P(C)$$

$$P(A) = P(B) + P(C)$$

$$P(B) = \frac{1}{2} P(B) + \frac{1}{4} P(A)$$

$$P(B) = \frac{1}{2} P(A)$$

$$P(C) = \frac{1}{2} P(C) + \frac{1}{4} P(A)$$

$$P(C) = \frac{1}{2} P(A)$$

But $P(A) + P(B) + P(C) = 1$

$$P(A) + \frac{1}{2} P(A) + \frac{1}{2} P(A) = 1 \qquad P(A) = \frac{1}{2} \qquad P(B) = P(C) = \frac{1}{4}$$

$$H_i = \sum_{j=A}^{C} P_{ij} \log \frac{1}{P_{ij}}$$

$$H_A = P_{AA} \log \frac{1}{P_{AA}} + P_{AB} \log \frac{1}{P_{AB}} + P_{AC} \log \frac{1}{P_{AC}} = \frac{1}{2} \log 2 + \frac{1}{4} \log 4 + \frac{1}{4} \log 4$$

$H_A = 1.5$ bits/MS

$H_B = \frac{1}{2} \log 2 + \frac{1}{4} \log 4 = 1$ bit/MS

$H_C = \frac{1}{2} \log 2 + \frac{1}{4} \log 4 = 1$ bit/MS

$$H = \sum P_i H_i = \frac{1}{2}(1.5) + \frac{1}{4}(1) + \frac{1}{4}(1)$$

$H = 1.25$ bits/MS

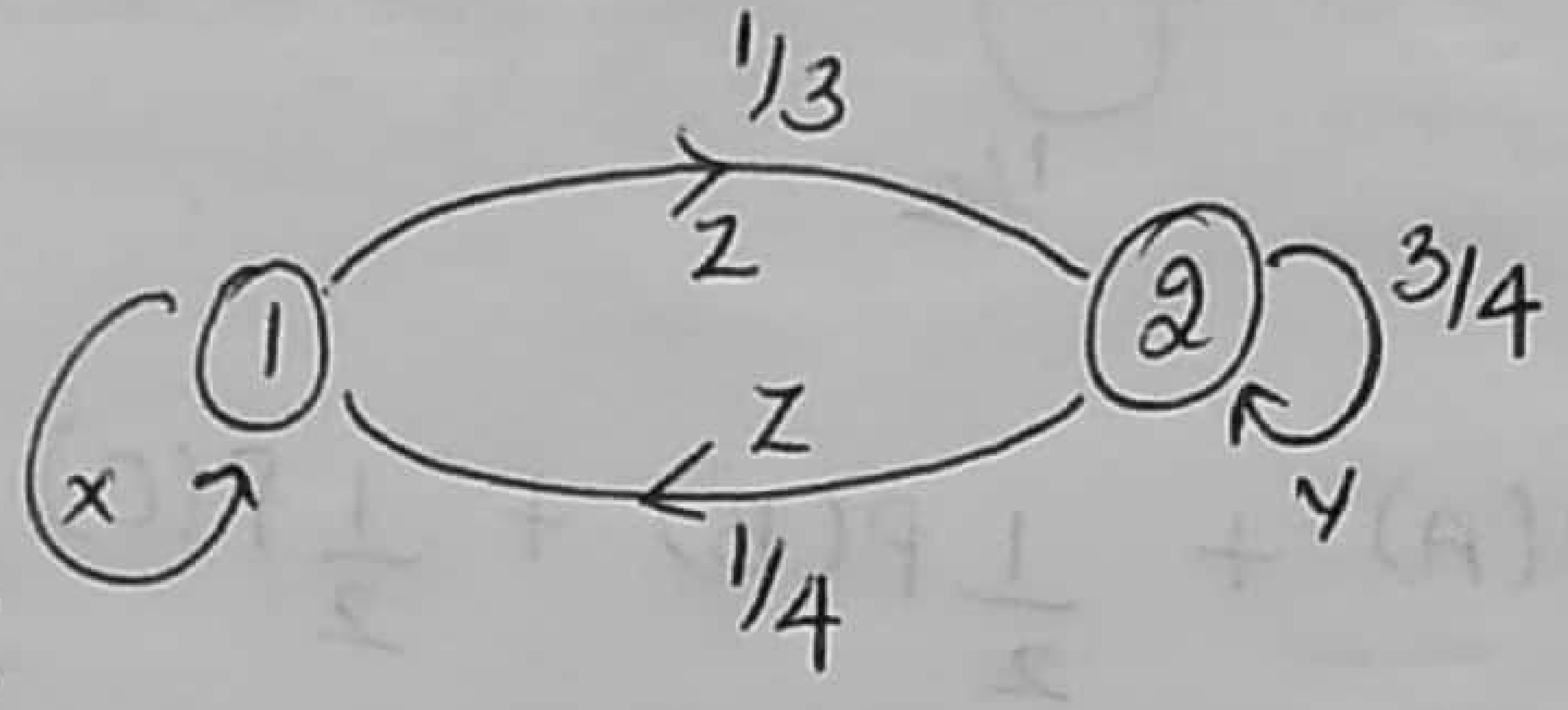
Assignment

2018

⑤ A state diagram of Markov source is shown in figure below.

i) Find the entropy of the source.

ii) Find G_1, G_2 and G_3 and verify that $G_1 > G_2 > G_3 > H$



2018

MODULE - 2

Source Coding

Coding :- Representing some information using code alphabet is called coding.

Properties of codes

1) Block code : A block code is a code which maps each of the symbols of the source alphabet S into some finite sequences of code symbols from the code alphabet X and each of these finite sequences is called code word.

Eg :- $S = \{S_1, S_2, S_3, S_4\}$ $X = \{0, 1\}$

Source symbol	Code - A
S_1	00
S_2	01
S_3	10
S_4	11

→ code words
↓
combination of code alphabet

No two source symbols have same code.

2) Non - singular code : A block code is said to be non-singular if and only if all the code words are distinct and easily distinguishable from one another.

Eg :-

Source symbol	Code - B
S_1	0
S_2	00
S_3	01
S_4	11

Second extension

$S_1 S_1$ - 00 $S_1 S_3$ - 001 $S_2 S_1$ - 000 $S_2 S_3$ - 0001

$S_1 S_2$ - 000 $S_1 S_4$ - 011 $S_2 S_2$ - 0000 $S_2 S_4$ - 0011

S_3S_1 - 010 S_3S_3 - 0101 S_4S_1 - 110 S_4S_3 - 1101 2018

S_3S_2 - 0100 S_3S_4 - 0111 S_4S_2 - 1100 S_4S_4 - 1111

For code A → code A' → second extension of code A

S_1S_2 - 0001 S_1S_3 - 0010 S_2S_1 - 0100 S_2S_2 - 0101

S_1S_1 - 0000 S_1S_4 - 0011 S_2S_3 - 0110 S_2S_4 - 0111

S_3S_1 - 1000 S_3S_3 - 1010 S_4S_1 - 1100 S_4S_3 - 1110

S_3S_2 - 1001 S_3S_4 - 1011 S_4S_2 - 1101 S_4S_4 - 1111

No codes are same. ∴ It is a non-singular code.

3) <u>Uniquely decodable code</u>: A block code is said to be uniquely decodable if and only if the n^{th} extension code words of the code is non-singular for every finite value of n.

Eg:- Consider code A and code B. [table].

Received sequence :- 001100

code A - $S_1S_4S_1$ [taken 2 bit at a time ∵ code A has 2 bits]

code B - $S_1S_1S_4S_1S_1$ | $S_2S_4S_2$ | $S_2S_4S_1S_1$ | $S_1S_1S_4S_2$ | S_1S_3

As by using code B we get more alternatives, code B is not uniquely decodable whereas code A is uniquely decodable.

4) <u>Instantaneous code</u>: A uniquely decodable code is said to be instantaneous if it is possible to recognise the end of any code word in any received sequence without refrencing any succeeding symbol i.e, there is no time delay in the process of decoding and decoding is done instantaneously as and when the symbols are arrived at the receiver.

Source symbols	code-C	code-D	code-E
S_1	00	0	0
S_2	01	10	01
S_3	10	110	011
S_4	11	1110	0111

20/8

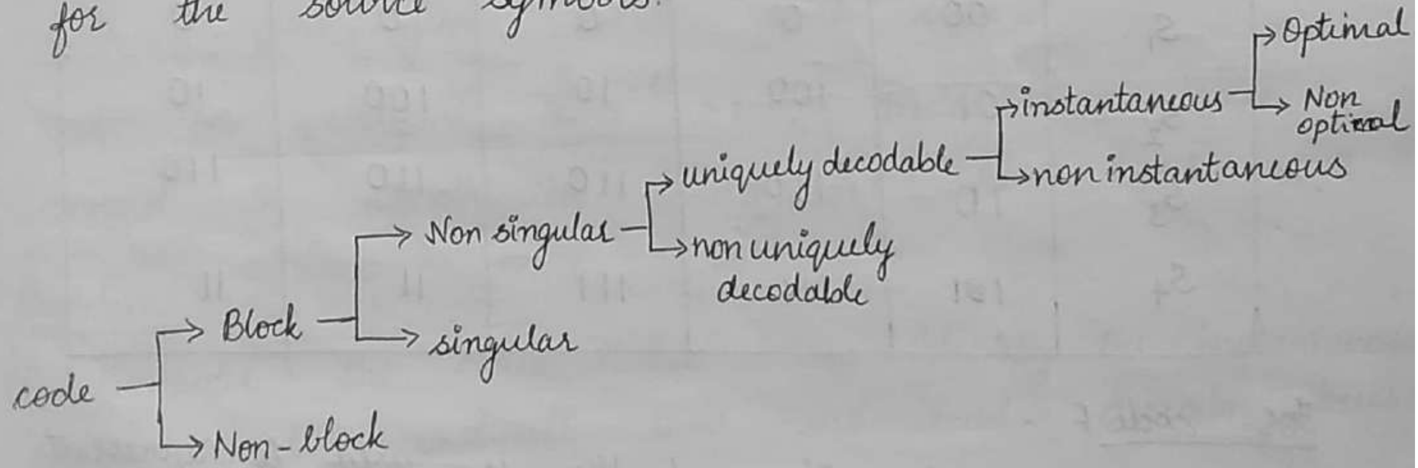
Received symbol 001100

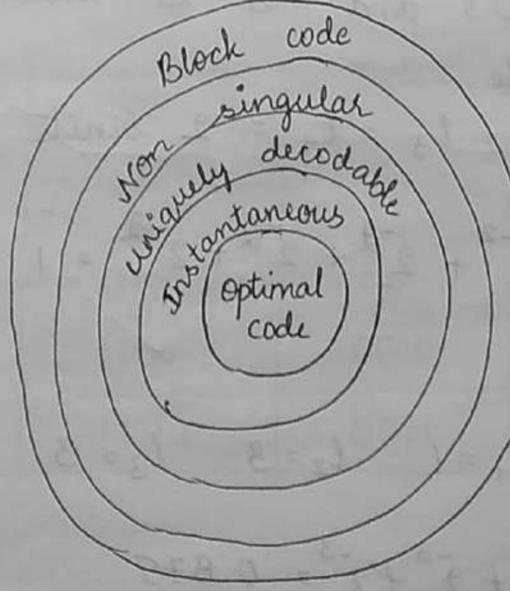
code C - $S_1 S_4 S_1$ } → Instantaneous code
code D - $S_1 S_1 S_3 S_1$ }
code E - $S_1 S_3 S_1 S_1$ → to decode using code E we have to wait for the arrival for the next bit.

In code E 0 is prefix of S_2, S_3, S_4
01 is a code and is prefix of S_3, S_4
011 is a code and is prefix of S_4
∴ code E is not instantaneous code.

No codeword should be the prefix of other code. Then it is called instantaneous code.

5) <u>Optimal codes</u> : An instantaneous code is said to be optimal code if it has minimum average length 'L' for a source with a given probability assignment for the source symbols.





22/8

Kraft - Inequality (Kraft - McMilani Inequality)

A neccessary and suffient condition for the existence of an instantaneous code with word lengths $L_1, L_2 \dots$ $\dots L_q$ is that $\boxed{\sum_{i=1}^{q} r^{-l_i} \leq 1}$

where r is the number of code alphabets
l is the length of each code word
↳ It is represented using binary units
binits

Only 0, 1 then $r = 2$ → binits

Only 0, 1, 2 then $r = 3$ → trinits (ternary units)

0, 1, 2, 3 then $r = 4$ → quaternary units

Eg:-

Source symbols	Code F	Code G	Code H	Code I	Code J
S_1	00	0	0	0	0
S_2	01	100	10	100	10
S_3	10	110	110	110	110
S_4	11	111	111	11	11

For code F :-

$r = 2$ ∵ only 0's and 1's is used to represent the code.

$q = 4$, $l_1 = l_2 = l_3 = l_4 = 2$ binits

$$\sum_{i=1}^{4} 2^{-l_i} = 2^{-2} + 2^{-2} + 2^{-2} + 2^{-2} = 1$$

For code G :-

$r = 2$ $q = 4$ $l_1 = 1$ $l_2 = 3$ $l_3 = 3$ $l_4 = 3$ binits

$$\sum_{i=1}^{4} 2^{-l_i} = 2^{-1} + 2^{-3} + 2^{-3} + 2^{-3} = 0.875$$

For code H :-

$r = 2 \quad q = 4 \quad l_1 = 1 \quad l_2 = 2 \quad l_3 = 3 \quad l_4 = 3$ binits

$$\sum_{i=1}^{4} 2^{-l_i} = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} = 1$$

For code I :-

$r = 2 \quad q = 4 \quad l_1 = 1 \quad l_2 = l_3 = 3 \quad l_4 = 2$ binits

$$\sum_{i=1}^{4} 2^{-l_i} = 2^{-1} + 2^{-3} + 2^{-3} + 2^{-2} = 1$$

For code J :-

$r = 2 \quad q = 4 \quad l_1 = 1 \quad l_2 = 2 \quad l_3 = 3 \quad l_4 = 2$ binits

$$\sum_{i=1}^{4} 2^{-l_i} = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-2} = 1.125$$

For code F, G, H, I ; $\sum_{i=1}^{q} r^{-l_i} \leq 1$ $\therefore$ we can say that instantaneous code can be constructed using the code lengths given under each code.

But for code J, $\sum_{i=1}^{q} r^{-l_i} > 1$ $\therefore$ Instantaneous code cannot be constructed using $l_1 = 1$, $l_2 = 2$, $l_3 = 3$, $l_4 = 2$.

Problems

① Consider the codes listed below. Identify the instantaneous codes and construct their individual decision trees.

Source symbols	Code K	Code L	Code M	Code N
S_1	0	0	0	00
S_2	10	01	01	01
S_3	110	001	011	10
S_4	1110	0010	110	110
S_5	1111	0011	111	111

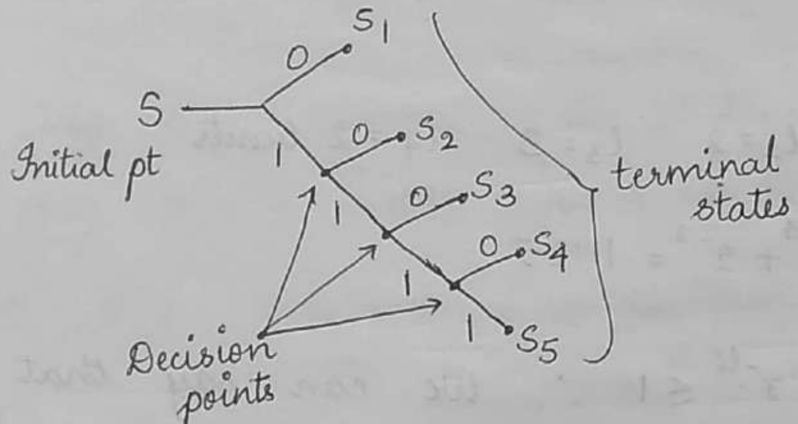
22/8

Solⁿ:- In code L & code M, code word 0 is a prefix of other code word. ∴ Code L and code M is non instantaneous code.

Code K and code N are instantaneous code.

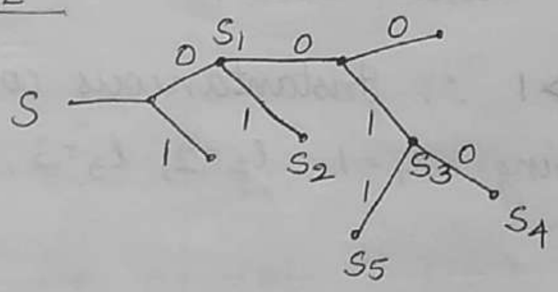
Code tree / Decision tree

Code K



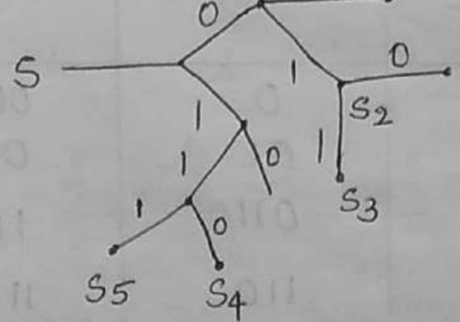
Terminal points / states are not acting as decision points
→ Instantaneous code

Code L

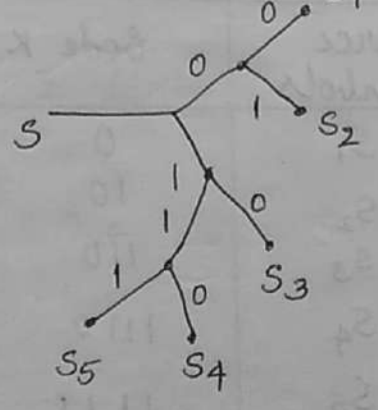


Terminal states (S_1, S_3) are acting as decision points.
→ Not instantaneous code

Code M



Code N



23/8

② The received code is 01101111111010. Decode this using code K, L, M & N

Sol:- Code K. 01101111111010

$S_1 S_3 S_5 S_4 S_2$

Code L S_2 } Can't be decoded ∵ not instantaneous.

Code M $S_3 S_3 S_5 S_4$ }

Code N $S_2 S_3 S_5 S_5 S_3 S_3$

③ Which of the following sets of word lengths specified in table below are acceptable for the existence of an instantaneous code given $X = \{0, 1, 2\}$?

No. of words of word length li			Word length
Code-P	Code-Q	Code-R	
2	2	1	1
1	2	4	2
2	2	6	3
4	3	0	4
1	1	0	5

Soln:- $r = 3$

In code P there are 2 codes with 1 trinit each

$$\sum_{i=1}^{q} r^{-l_i} = \sum_{i=1}^{10} 3^{-l_i}$$

$$= 2(3^{-1}) + 1(3^{-2}) + 2(3^{-3}) + 4(3^{-4}) + 1(3^{-5})$$

= ~~1.127~~ $0.9053 < 1$

Code Q $$\sum_{i=1}^{q} r^{-l_i} = \sum_{i=1}^{10} 3^{-l_i} = 2(3^{-1}) + 2(3^{-2}) + 2(3^{-3}) + 3(3^{-4}) + 3^{-5}$$

$$= 1.004 > 1$$

Code R $$\sum_{i=1}^{q} r^{-l_i} = \sum_{i=1}^{11} 3^{-l_i} = 3^{-1} + 4(3^{-2}) + 6(3^{-3}) = 1$$

28/8

Code Efficiency and redundancy

Average length of a code

$$L = \sum_{i=1}^{q} P_i l_i \text{ binits/MS}$$

$$H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} \text{ bits/MS}$$

$L \geq H(S) \Rightarrow$ binary

$L \geq H_r(S) \Rightarrow$ r-ary codes

$$H_r(S) = \frac{H(S)}{\log_2 r}$$

$$\eta_c = \frac{H(S)}{L} \text{ or } \eta_c = \frac{H_r(S)}{L}$$

Redundancy $R_{\eta_c} = 1 - \eta_c$

Problems

① A source having an alphabet $S = \{S_1\ S_2\ S_3\ S_4\ S_5\}$ produces these symbols with respective probabilities of $P = \{\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{9}, \frac{1}{18}\}$. i) When these symbols are coded as code N, find efficiency and redundancy.
ii) When these symbols are coded as code K, find efficiency and redundancy.

Sol :- $H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} = \sum_{i=1}^{5} P_i \log \frac{1}{P_i}$

$$= \frac{1}{2}\log 2 + 2 \times \frac{1}{6}\log 6 + \frac{1}{9}\log 9 + \frac{1}{18}\log 18$$

$H(S) = 1.946$ bits/MS

i) code N $\quad \eta_c = \frac{H(S)}{L}$

$$L = \sum_{i=1}^{q} P_i l_i = \sum_{i=1}^{5} P_i l_i$$

23/8

$$L = \frac{1}{2}(2) + \frac{1}{6}(2) + \frac{1}{6}(2) + \frac{1}{9}(3) + \frac{1}{18}(3)$$

$L = 2.167$ binits/MS

Code * $\quad \eta_c = \dfrac{H(S)}{L} = \dfrac{1.946}{2.167} = 0.898 \times 100 = 89.8\%.$

$$R_{\eta_c} = 1 - \eta_c = 1 - 0.898 = 0.102 = 10.2\%.$$

ii) Code K

$$L = \sum_{i=1}^{5} P_i l_i = \frac{1}{2}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{9}(4) + \frac{1}{18}(4)$$

$$L = 2$$

$$\eta_c = \frac{H(S)}{L} = \frac{1.946}{2} = 0.973 = 97.3\%.$$

$$R_{\eta_c} = 1 - \eta_c = 1 - 0.973 = 0.027 = 2.7\%.$$

Shannon's encoding algorithm

Steps

1. List the source symbols in the order of non increasing probabilities. Given $S = \{S_1, S_2 \dots S_q\}$ with probabilities $P = \{P_1, P_2 \dots P_q\}$; $P_1 \geq P_2 \geq P_3 \geq \dots \geq P_q$.
2. Compute the sequences $\alpha_1 = 0$; $\alpha_2 = P_1 = P_1 + \alpha_1$; $\alpha_3 = P_2 + P_1 = P_2 + \alpha_2$; $\alpha_4 = P_3 + P_2 + P_1 = P_3 + \alpha_3 \dots$ $\alpha_{q+1} = P_q + P_{q-1} + \dots + P_1 = P_q + \alpha_q$
3. Determine the smallest integer value of l_i using the inequality $2^{l_i} \geq \frac{1}{P_i}$ for all $i = 1, 2 \dots q$
4. Expand the decimal number α_i in binary form upto l_i places neglecting expansion beyond l_i places.
5. Remove the binary point to get the desired code

26/8

Problems

① Apply Shannon's encoding algorithm to the following set of messages and obtain code efficiency and redundancy.

m_1	m_2	m_3	m_4	m_5
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{8}$

Sol :-

m_1	m_2	m_3	m_4	m_5
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{8}$
$\frac{2}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{96}{16}$

Step 1 :

m_5	m_4	m_3	m_1	m_2
$\frac{6}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

Step 2 :- $\alpha_1 = 0$ $q = 5$

$$\alpha_2 = P_1 + \alpha_1 = \frac{6}{16} = 0.375$$

$$\alpha_3 = P_2 + \alpha_2 = \frac{4}{16} + \frac{6}{16} = \frac{5}{8} = 0.625$$

$$\alpha_4 = P_3 + \alpha_3 = \frac{3}{16} + \frac{5}{8} = \frac{13}{16} = 0.8125$$

$$\alpha_5 = P_4 + \alpha_4 = \frac{2}{16} + \frac{13}{16} = \frac{15}{16} = 0.9375$$

$$\alpha_{q+1} = \alpha_6 = P_5 + \alpha_5 = \frac{1}{16} + \frac{15}{16} = 1$$

Step 3 :- $2^{l_i} \geq \frac{1}{P_i}$

$$i=1,\ 2^{l_1} \geq \frac{1}{P_1} = \frac{1}{6/16} = \cancel{8}\ \frac{16}{6} = 2.66$$

$$i=2,\ 2^{l_2} \geq \frac{1}{P_2} = \frac{1}{4/16} = 4 \qquad 2^{l_2} \geq 4$$

$$i=3,\ 2^{l_3} = \frac{1}{P_3} = \frac{1}{3/16} = \frac{16}{3} \qquad 2^{l_3} \geq 5.33$$

26|8

$2^{l_4} = \frac{1}{P_4} = \frac{1}{2/16} = 8$ $\quad 2^{l_4} \geq 8$

$2^{l_5} = \frac{1}{P_5} = \frac{1}{1/16} = 16$ $\quad 2^{l_5} \geq 16$

$2^{l_1} \geq 2.66$

$l_1 \geq \log_2 2.66$

$l_1 = 1.41$

$\boxed{l_1 = 2}$

$2^{l_2} \geq 4$

$l_2 \geq \log_2 2$

$\boxed{l_2 = 2}$

$2^{l_3} \geq 5.33$

$l_3 \geq \log_2 5.33$

$l_3 = 2.41$

$\boxed{l_3 = 3}$

$2^{l_4} \geq 8$

$l_4 \geq \log_2 8$

$\boxed{l_4 = 3}$

$2^{l_5} \geq 16$

$l_5 \geq \log_2 16$

$\boxed{l_5 = 4}$

Step 4 :

$\alpha_1 = 0$ and as $l_1 = 2$ (length of code = 2)

it is coded as 00

$\alpha_1 = 0$ 00

$\alpha_2 = (0.375)_{10}$

$0.375 \times 2 = 0.75$ with carry 0 ↓

$0.75 \times 2 = 1.5 \rightarrow 1$

$0.5 \times 2 = 1 \rightarrow 1$

$\alpha_2 = (0.011)_2 = 01 \rightarrow$ the 2 bits after decimal point is the code as the length of code = 2 (l_2)

$\alpha_3 = (0.625)_{10}$

$0.625 \times 2 = 1.25 \rightarrow 1$

$0.25 \times 2 = 0.5 \rightarrow 0$

$0.5 \times 2 = 1 \rightarrow 1$

$\alpha_3 = (0.101)_2 = 101$

$l_3 = 3$

$\alpha_4 = (0.8125)_{10}$

$0.8125 \times 2 = 1.625 \rightarrow 1$

$0.625 \times 2 = 1.25 \rightarrow 1$

$0.25 \times 2 = 0.5 \rightarrow 0$

$0.5 \times 2 = 1 \rightarrow 1$

$\alpha_4 = (0.1101)_2 = 110$

$l_4 = 3$

$\alpha_5 = 0.9375 = (0.1111)_2 \quad l_5 = 4$

<u>Step 5</u> :- $\alpha_1 = 0, \quad l_1 = 2 \quad \therefore$ code for $m_5 = 00$

$\alpha_2 = (0.011)_2, \quad l_2 = 2 \quad \therefore$ code for $m_4 = 01$

$\alpha_3 = (0.101)_2, \quad l_3 = 3 \quad \therefore$ code for $m_3 = 101$

$\alpha_4 = (0.1101)_2, \quad l_4 = 3 \quad \therefore$ code for $m_1 = 110$

$\alpha_5 = (0.1111)_2, \quad l_5 = 4 \quad \therefore$ code for $m_2 = 1111$

Symbols	Probabilities	length	code
m_5	6/16	2	00
m_4	4/16	2	01
m_3	3/16	3	101
m_1	2/16	3	110
m_2	1/16	4	1111

$$H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} = \sum_{i=1}^{14} P_i \log \frac{1}{P_i}$$

$$= 2\left(\frac{6}{16}\log\frac{16}{6}\right) + 2\left(\frac{4}{16}\log\frac{16}{4}\right) + 3\left(\frac{3}{16}\log\frac{16}{3}\right) + 3\left(\frac{2}{16}\log\frac{16}{2}\right)$$
$$+ 4\left(\frac{1}{16}\log 16\right)$$

$H(S) = \overset{0.635}{8.545}$ bits/message symbol $= 2.108$ bits/MS

$$L = \sum_{i=1}^{q} P_i l_i = \sum_{i=1}^{5} P_i l_i$$

$$= 2\left(\frac{6}{16}\right) + 2\left(\frac{4}{16}\right) + 3\left(\frac{3}{16}\right) + 3\left(\frac{2}{16}\right) + 4\left(\frac{1}{16}\right)$$

$= 2.4375$ binits/MS

$$\eta_c = \frac{H(S)}{L} = \frac{2.108}{2.4375} = 0.8648 = 86.5\%$$

$R_{\eta_c} = 1 - \eta_c = 1 - 0.8648 = 0.1352$

$= 13.52\ \%$

② Apply Shannon's encoding (binary) algorithm to the following messages

S_1	S_2	S_3
0.5	0.3	0.2

i) Find code efficiency and redundancy

ii) If the same technique is applied to the 2nd order extension of this source, how much will the code efficiency be improved?

Solⁿ:- i) S_1 S_2 S_3

Step-1 0.5 0.3 0.2

$\frac{1}{2}$ $\frac{3}{10}$ $\frac{1}{5}$

Step 2 $\alpha_1 = 0$ $q = 3$

$\alpha_2 = P_1 + \alpha_1 = 0.5$

$\alpha_3 = P_2 + \alpha_2 = 0.3 + 0.5 = 0.8$

$\alpha_4 = P_3 + \alpha_3 = 0.2 + 0.8 = 1$

Step 3 $2^{l_i} \geq \frac{1}{P_i}$

$i = 1,\ 2^{l_1} \geq \frac{1}{P_1} \geq \frac{1}{1/2} \geq 2$

$2^{l_1} \geq 2$ $l_1 \geq \log 2$ $l_1 = 1$

$i = 2,\ 2^{l_2} \geq \frac{1}{P_2} \geq \frac{1}{0.3} \geq 3.33$

$2^{l_2} \geq 3.33$ $l_2 = 1.73$ $l_2 = 2$

$i = 3,\ 2^{l_3} \geq \frac{1}{P_3} \geq \frac{1}{0.2} \geq 5$

$2^{l_3} \geq 5$ $l_3 = 2.32$ $l_3 = 3$

Step 4 $\alpha_1 = 0 \Rightarrow 0$

$\alpha_2 = (0.5)_{10} = (.1)_2 \quad l_1 = 1$

$0.5 \times 2 = 1$

$\alpha_2 = (.10)_2 \Rightarrow 10 \quad l_2 = 2$

$\alpha_3 = (0.8)_{10}$

$0.8 \times 2 = 1.6 \rightarrow 1$ $\alpha_3 = (0.11001\ldots)_2$
$0.6 \times 2 = 1.2 \rightarrow 1$
$0.2 \times 2 = 0.4 \rightarrow 0$ $\alpha_3 \Rightarrow 110 \quad l_3 = 3$
$0.4 \times 2 = 0.8 \rightarrow 0$
$0.8 \times 2 = 1.6 \rightarrow 1$

~~$\alpha_4 = 1$~~ ~~(0.1111)~~ ~~$\Rightarrow 1111$~~ ~~$l_4 = 4$~~

Step 5 $\alpha_1 = 0, \; l_1 = 1 \; \therefore$ code for $S_1 = 0$
$\alpha_2 = (0.10)_2, \; l_2 = 2 \; \therefore$ code for $S_2 = 10$
$\alpha_3 = (0.11001..)_2, \; l_3 = 3 \; \therefore$ code for $S_3 = 110$

Symbols	Probabilities	length	code
S_1	0.5	1	0
S_2	0.3	2	10
S_3	0.2	3	110

$$H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} = \sum_{i=1}^{3} P_i \log \frac{1}{P_i}$$

$$= 0.5 \log \frac{1}{0.5} + 0.3 \log \frac{1}{0.3} + 0.2 \log \frac{1}{0.2}$$

$$H(S) = 1.485 \text{ bits/MS}$$

$$L = \sum_{i=1}^{q} l_i P_i = \sum_{i=1}^{3} L_i P_i = 0.5 + 2(0.3) + 3(0.2)$$

$$= 1.7 \text{ binits/MS}$$

$$\eta_c = \frac{H(S)}{L} = \frac{1.485}{1.7} = 0.873 = 87.3\%$$

$$R_{\eta_c} = 1 - \eta_c = 0.127 = 12.7\%$$

26/8

ii) Second order extension

$S_1S_1 \rightarrow 0.5 \times 0.5 = 0.25$

$S_1S_2 \rightarrow 0.5 \times 0.3 = 0.15$

$S_1S_3 \rightarrow 0.5 \times 0.2 = 0.1$

$S_2S_1 \rightarrow 0.3 \times 0.5 = 0.15$

$S_2S_2 \rightarrow 0.3 \times 0.3 = 0.09$

$S_2S_3 \Rightarrow 0.3 \times 0.2 = 0.06$

$S_3S_1 \rightarrow 0.2 \times 0.5 = 0.1$

$S_3S_2 \rightarrow 0.2 \times 0.3 = 0.06$

$S_3S_3 \rightarrow 0.2 \times 0.2 = 0.04$

Step 1 :-

S_1S_1	S_1S_2	S_2S_1	S_1S_3	S_3S_1	S_2S_2	S_2S_3	S_3S_2	S_3S_3
0.25	0.15	0.15	0.1	0.1	0.09	0.06	0.06	0.04

Step 2 :- $\alpha_1 = 0 \qquad q = 9$

$\alpha_2 = P_1 + \alpha_1 = 0.25$

$\alpha_3 = P_2 + \alpha_2 = 0.15 + 0.25 = 0.4$

$\alpha_4 = P_3 + \alpha_3 = 0.15 + 0.4 = 0.55$

$\alpha_5 = P_4 + \alpha_4 = 0.1 + 0.55 = 0.65$

$\alpha_6 = P_5 + \alpha_5 = 0.1 + 0.65 = 0.75$

$\alpha_7 = P_6 + \alpha_6 = 0.09 + 0.75 = 0.84$

$\alpha_8 = P_7 + \alpha_7 = 0.06 + 0.84 = 0.9$

$\alpha_9 = P_8 + \alpha_8 = 0.06 + 0.9 = 0.96$

$\alpha_{10} = P_9 + \alpha_9 = 0.04 + 0.96 = 1$

Step 3 :- $2^{l_i} \geq \frac{1}{P_i}$

$i=1 ; \quad 2^{l_1} \geq \frac{1}{0.25} \geq 4 \qquad l_1 = 2$

$i=2 , \quad 2^{l_2} \geq \frac{1}{0.15} \geq 6.66 \qquad l_2 = 2.73 \quad l_2 = 3$

$i=3 , \quad 2^{l_3} \geq \frac{1}{0.15} \geq 6.66 \qquad l_3 = 3$

$i=4 , \quad 2^{l_4} \geq \frac{1}{0.1} \geq 10 \qquad l_4 = 4$

$i=5 , \quad 2^{l_5} \geq \frac{1}{0.1} \geq 10 \qquad l_5 = 4$

26/8

$i = 6, \quad 2^{l_6} \geq \frac{1}{0.09} \geq 11.11$ $\qquad l_6 = 3.47 = 4$

$i = 7, \quad 2^{l_7} \geq \frac{1}{0.06} \geq 16.66$ $\qquad l_7 = 4.05 = 5$

$i = 8, \quad 2^{l_8} \geq \frac{1}{0.06} \geq 16.66$ $\qquad l_8 = 4.05 = 5$

$i = 9, \quad 2^{l_9} \geq \frac{1}{0.04} \geq 25$ $\qquad l_9 = 4.64 = 5$

Step 4 :- $\alpha_1 = 0 \Rightarrow 00$ $\qquad l_1 = 2$

$\alpha_2 = (0.25)_{10}$ $\qquad \alpha_2 = (0.01)_2 \Rightarrow 010$

$0.25 \times 2 = 0.5 \rightarrow 0$

$0.5 \times 2 = 1 \rightarrow 1$ $\qquad l_2 = 3$

$\alpha_3 = (0.4)_{10}$

$0.4 \times 2 = 0.8 \rightarrow 0$ $\qquad \alpha_3 = (0.0110)_2 \Rightarrow 011$

$0.8 \times 2 = 1.6 \rightarrow 1$

$0.6 \times 2 = 1.2 \rightarrow 1$ $\qquad l_3 = 3$

$0.2 \times 2 = 0.4 \rightarrow 0$

$\alpha_4 = (0.55)_{10}$ $\qquad l_4 = 4$

$0.55 \times 2 = 1.1 \rightarrow 1$

$0.1 \times 2 = 0.2 \rightarrow 0$ $\qquad \alpha_4 = (0.1000)_2 \Rightarrow 1000$

$0.2 \times 2 = 0.4 \rightarrow 0$

$0.4 \times 2 = 0.8 \rightarrow 0$

$\alpha_5 = (0.65)_{10}$ $\qquad l_5 = 4$

$0.65 \times 2 = 1.3 \rightarrow 1$

$0.3 \times 2 = 0.6 \rightarrow 0$ $\qquad \alpha_5 = (0.1010)_2 \Rightarrow 1010$

$0.6 \times 2 = 1.2 \rightarrow 1$

$0.2 \times 2 = 0.4 \rightarrow 0$

$\alpha_6 = (0.75)_{10}$

$0.75 \times 2 = 1.5 \rightarrow 1$

$0.5 \times 2 = 1 \rightarrow 1$

$l_6 = 4$

$\alpha_6 = (0.1100)_2 \Rightarrow 1100$

$\alpha_7 = (0.84)_{10}$

$0.84 \times 2 = 1.68 \rightarrow 1$

$0.68 \times 2 = 1.36 \rightarrow 1$

$0.36 \times 2 = 0.72 \rightarrow 0$

$0.72 \times 2 = 1.44 \rightarrow 1$

$0.44 \times 2 = 0.88 \rightarrow 0$

$l_7 = 5$

$\alpha_7 = (0.11010)_2 \Rightarrow 11010$

$\alpha_8 = (0.9)_{10}$

$0.9 \times 2 = 1.8 \rightarrow 1$

$0.8 \times 2 = 1.6 \rightarrow 1$

$0.6 \times 2 = 1.2 \rightarrow 1$

$0.2 \times 2 = 0.4 \rightarrow 0$

$0.4 \times 2 = 0.8 \rightarrow 0$

$l_8 = 5$

$\alpha_8 = (0.11100)_2 \Rightarrow 11100$

$\alpha_9 = (0.96)_{10}$

$0.96 \times 2 = 1.92 \rightarrow 1$

$0.92 \times 2 = 1.84 \rightarrow 1$

$0.84 \times 2 = 1.68 \rightarrow 1$

$0.68 \times 2 = 1.36 \rightarrow 1$

$0.36 \times 2 = 0.72 \rightarrow 0$

$l_9 = 5$

$\alpha_9 = (0.11110)_2 \Rightarrow 11110$

Symbol	probabilities	length	code
S_1S_2	0.25	2	00
S_1S_2	0.15	3	010
S_2S_1	0.15	3	011
S_1S_3	0.1	4	1000
S_3S_1	0.1	4	1010

26/8

symbol	probabilities	length	code
S_2S_2	0.09	4	1100
S_2S_3	0.06	5	11010
S_3S_2	0.06	5	11100
S_3S_3	0.04	5	11110

$$H(S^2) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} = \sum_{i=1}^{9} P_i \log \frac{1}{P_i}$$

$$= 0.25 \log \frac{1}{0.25} + (0.15)2 \log \frac{1}{0.15} + 2(0.1) \log \frac{1}{0.1} + 0.09 \log \frac{1}{0.09}$$

$$+ 2(0.06) \log \frac{1}{0.06} + 0.04 \log \frac{1}{0.04}$$

$$= 2.97 \text{ bits/MS}$$

$$L = \sum_{i=1}^{q} l_i P_i = \sum_{i=1}^{9} P_i l_i$$

$$= 0.25 \times 2 + (0.15)(3)(2) + 2(0.1)(4) + 4(0.09) + 5(0.04)$$

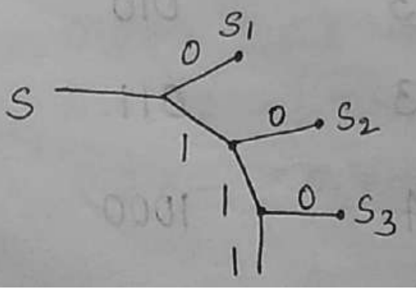
$$+ 2(0.06)(5)$$

$$= 3.36 \text{ binits/MS}$$

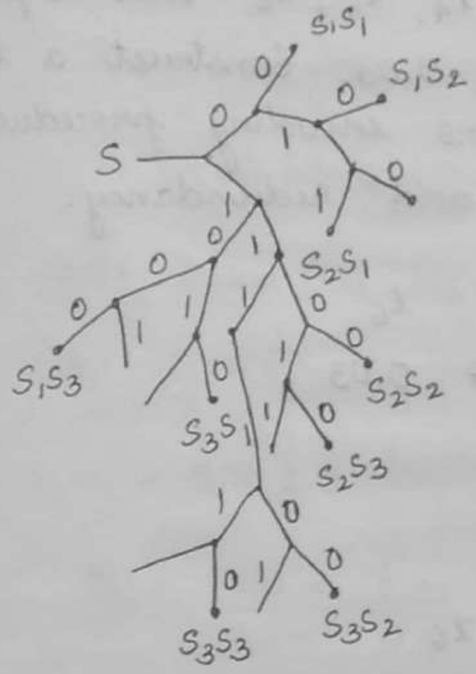
$$\eta_c = \frac{H(S)}{L} = \frac{2.97}{3.36} = 88.39\%$$

$$R_{\eta_c} = 1 - \eta_c = 1 - 0.8839 = 0.116 = 11.6\%$$

Code tree for S_1, S_2, S_3



Code tree for second extension



Shannon - Fano encoding algorithm

Steps

① The symbols are arranged according to non-increasing probabilities ② The symbols are divided into 2 groups so that sum of probabilities in each group is approximately equal.

③ All the symbols in I grp are designated by 1 and II grp by 0.

④ The I grp is again sub-divided into 2 sub-grps such that each sub-grp probabilities are approximately same.

⑤ All the symbols in the I sub grp are designated by 1 and II sub grp by 0.

⑥ The II group is sub-divided into 2 more sub-groups and step ⑤ is repeated.

⑦ This process is continued till further sub-division is impossible.

Problems

① Given the messages $x_1, x_2, x_3, x_4, x_5, x_6$ with respective probabilities 0.4, 0.2, 0.2, 0.1, 0.07, 0.03. Construct a binary code by applying Shannon-Fano encoding procedure. Determine the code efficiency and redundancy.

Sol :- Step 1

x_1	x_2	x_3	x_4	x_5	x_6
0.4	0.2	0.2	0.1	0.07	0.03

Step 2 I grp

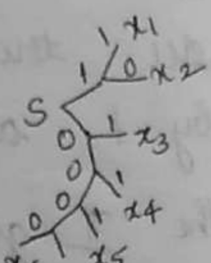
x_1	x_2
0.4	0.6

II grp

x_3	x_4	x_5	x_6
0.2	0.1	0.07	0.03.

I	x_1 0.4	1	x_1 0.4	1				
	x_2 0.2	1	x_2 0.2	0				
	x_3 0.2	0	x_3 0.2	1				
II	x_4 0.1	0	x_4 0.1	0	x_4 0.1	1		
	x_5 0.07	0	x_5 0.07	0	x_5 0.07	0	x_5 0.07	1
	x_6 0.03	0	x_6 0.03	0	x_6 0.03	0	x_6 0.03	0

Symbol	Probabilities	Code	Length
x_1	0.4	11	2
x_2	0.2	10	2
x_3	0.2	01	2
x_4	0.1	001	3
x_5	0.07	0001	4
x_6	0.03	0000	4



$H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} = \sum_{i=1}^{6} P_i \log \frac{1}{P_i}$

$= 0.4 \log \frac{1}{0.4} + 2(0.2) \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1} + 0.07 \log \frac{1}{0.07}$
$+ 0.03 \log \frac{1}{0.03}$

$H(S) = 2.21$ bits / MS

$L = \sum_{i=1}^{6} l_i P_i = 2 \times 0.4 + 2(0.2+0.2) + 3 \times 0.1 + 4(0.07+0.03)$

$=$ ~~5.9~~ 2.3 binits / MS

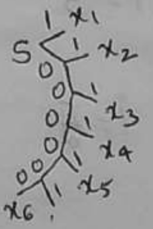
$\eta_c = \frac{H(S)}{L} = \frac{2.21}{2.3} = 0.37608 =$ ~~37.4%~~ 96.08%

$R\eta_c = 1 - \eta_c =$ ~~1-0.0392~~ $=$ ~~0.626~~ $=$ ~~62.6%~~
$1 - 0.9608 = 0.0392 = 3.92\%$

Alternative

x_1	0.4	1												
x_2	0.2	0	x_2	0.2	1									
x_3	0.2	0	x_3	0.2	0	x_3	0.2	1						
x_4	0.1	0	x_4	0.1	0	x_4	0.1	0	x_4	0.1	1			
x_5	0.07	0	x_5	0.07	0	x_5	0.07	0	x_5	0.07	0	x_5	0.07	1
x_6	0.03	0	x_6	0.03	0	x_6	0.03	0	x_6	0.03	0	x_6	0.03	0

Symbol	Probabilities	Code	length
x_1	0.4	1	1
x_2	0.2	01	2
x_3	0.2	001	3
x_4	0.1	0001	4
x_5	0.07	00001	5
x_6	0.03	00000	5



$L = 1 \times 0.4 + 2 \times 0.2 + 3 \times 0.2 + 4 \times 0.1 + 5 \times 0.07 + 5 \times 0.03$

$L = 2.3$ binits / MS

27/8

$H(S) = 2.21$ bits /ms

$$\eta_c = \frac{H(S)}{L} = \frac{2.21}{2.3} = 0.9608 = 96.08\%.$$

$$R_{\eta_c} = 1 - \eta_c = 1 - 0.9608 = 0.0392 = 3.92\%.$$

30/8

Previous problem solve using Shannon's encoding algorithm

Step 1 :

	x1	x2	x3	x4	x5	x6
	0.4	0.2	0.2	0.1	0.07	0.03

Step 2 :- $\alpha_1 = 0 \quad q = 6$

$\alpha_2 = P_1 + \alpha_1 = 0.4$

$\alpha_3 = P_2 + \alpha_2 = 0.6$

$\alpha_4 = P_3 + \alpha_3 = 0.8$

$\alpha_5 = P_4 + \alpha_4 = 0.9$

$\alpha_6 = P_5 + \alpha_5 = 0.97$

$\alpha_7 = P_6 + \alpha_6 = 1$

Step 3 :- $2^{l_i} \geq \frac{1}{P_i}$

$i = 1 \quad 2^{l_1} \geq \frac{1}{0.4} \geq 2.5 \quad l_1 = 2$

$i = 2 \quad 2^{l_2} \geq \frac{1}{0.2} \geq 5 \quad l_2 = 3$

$i = 3 \quad 2^{l_3} \geq \frac{1}{0.2} \geq 5 \quad l_3 = 3$

$i = 5 \quad 2^{l_5} \geq \frac{1}{0.07} \geq 14.29 \quad l_5 = 4$

$i = 6 \quad 2^{l_6} \geq \frac{1}{0.03} \geq 33.33 \quad l_6 = 6$

$i = 4 \quad 2^{l_4} \geq \frac{1}{0.1} \geq 10 \quad l_4 = 4$

Scanned with CamScanner

Step 4 :- $\alpha_1 = 0$ $l_1 = 2 \Rightarrow 00$

$\alpha_2 = 0.4$ $l_2 = 3 \Rightarrow (0.0110)$

$0.4 \times 2 = 0.8 \rightarrow 0$ $= 011$

$0.8 \times 2 = 1.6 \rightarrow 1$

$0.6 \times 2 = 1.2 \rightarrow 1$

$0.2 \times 2 = 0.4 \rightarrow 0$

$\alpha_3 = 0.6$ $\alpha_3 = (0.1000)_2$ $l_3 = 3$

$0.6 \times 2 = 1.2 \Rightarrow 1$ $\Rightarrow 100$

$0.2 \times 2 = 0.4 \rightarrow 0$

$0.4 \times 2 = 0.8 \rightarrow 0$

$0.8 \times 2 = 1.6 \rightarrow 0$

$\alpha_4 = 0.8$ $\alpha_4 = (0.1100)_2$ $l_4 = 4$

$0.8 \times 2 = 1.6 \rightarrow 1$ $\Rightarrow 1100$

$0.6 \times 2 = 1.2 \rightarrow 1$

$0.2 \times 2 = 0.4 \rightarrow 0$

$0.4 \times 2 = 0.8 \rightarrow 0$

$\alpha_5 = 0.9$ $\alpha_5 = (0.11100)_2$ $l_5 = 4$

$0.9 \times 2 = 1.8 \rightarrow 1$ $\Rightarrow 1110$

$0.8 \times 2 = 1.6 \rightarrow 1$

$0.6 \times 2 = 1.2 \rightarrow 1$

$0.2 \times 2 = 0.4 \rightarrow 0$

$0.4 \times 2 = 0.8 \rightarrow 0$

$\alpha_6 = 0.97$ $\alpha_6 = (0.111110)_2$ $l_6 = 6$

$0.97 \times 2 = 1.94 \rightarrow 1$ $\Rightarrow 111110$

$0.94 \times 2 = 1.88 \rightarrow 1$

$0.88 \times 2 = 1.76 \rightarrow 1$

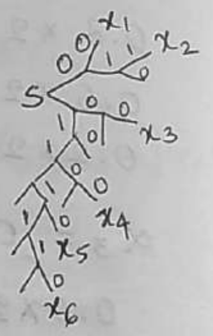
$0.76 \times 2 = 1.52 \rightarrow 1$

$0.52 \times 2 = 1.04 \rightarrow 1$

$0.04 \times 2 = 0.08 \rightarrow 0$

30/8

Symbol	probabilities	length	code
x_1	0.4	2	00
x_2	0.2	3	011
x_3	0.2	3	100
x_4	0.1	4	1100
x_5	0.07	4	1110
x_6	0.03	6	111110



$$H(S) = \sum_{i=1}^{6} P_i \log \frac{1}{P_i}$$

$$= 0.4 \log \frac{1}{0.4} + 2\times 0.2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1} + 0.07 \log \frac{1}{0.07} + 0.03 \log \frac{1}{0.03}$$

$H(S) = 2.21$ bits/MS

$$L = \sum_{i=1}^{6} l_i P_i = 2(0.4) + 3(0.2) + 3(0.2) + 4(0.1) + 4(0.07) + 6(0.03)$$

$L = 2.86$ binits/ms

$$\eta_c = \frac{H(S)}{L} = \frac{2.21}{2.86} = 0.7727 = 77.27\%$$

$$R_{\eta_c} = 1 - \eta_c = 0.2273 = 22.73\%$$

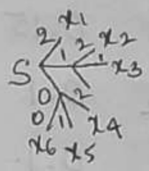
For the previous problem apply Shannon Fano ternary algorithm

x_1	0.4	2
x_2	0.2	1
x_3	0.2	1
x_4	0.1	0
x_5	0.07	0
x_6	0.03	0

x_2	0.2	2
x_3	0.2	1
I	0	0
x_4	0.1	2
x_5	0.07	1
x_6	0.03	0

30/8

Symbol	Probabilities	length	code
x_1	0.4	1	2
x_2	0.2	2	12
x_3	0.2	2	11
x_4	0.1	2	02
x_5	0.07	2	01
x_6	0.03	2	00



$H(S) = 2.21$ bits/MS

$L = \sum_{i=1}^{6} P_i l_i = 0.4 \times 1 + 2(0.2+0.2+0.1+0.07+0.03)$

$L = 1.6$ trinits/MS

~~$\eta_c = \frac{H(S)}{L} = \frac{2.21}{1.6} = 1.3812 = 138.12\%$~~

$R\eta_c$

$r = 3$

$H_r(S) = \frac{H(S)}{\log_2 r}$ $\qquad H_r(S) = \frac{2.21}{\log_2 3}$

$H_r(S) = 1.39$ ~~bits~~ trinits/MS

$\eta_c = \frac{H_r(S)}{L} = \frac{1.39}{1.6} = 0.8687 = 86.87\%$

$R\eta_c = 1 - 0.8714 = 0.1325 = 13.125\%$

$\eta_c = 86.87\%$ $\qquad R\eta_c = 13.125\%$

② We have the source symbols $S = \{S_1, S_2, S_3\}$ with probabilities $\{0.5, 0.2, 0.3\}$. For the II extension of source find efficiency and redundancy by i) applying Shannon Fano encoding algorithm.

ii) Apply Shannon Fano ternary algorithm

Solⁿ :-

$S_1S_2 = 0.25$	$S_2S_1 = 0.1$	$S_3S_1 = 0.15$
$S_1S_2 = 0.1$	$S_2S_2 = 0.04$	$S_3S_2 = 0.06$
$S_1S_3 = 0.15$	$S_2S_3 = 0.06$	$S_3S_3 = 0.09$

i)

S_1S_1	0.25	1	S_1S_1	0.25	1					
S_1S_3	0.15	1	S_1S_3	0.15	0	S_1S_3	0.15 → 1			
S_3S_1	0.15	1	S_3S_1	0.15	0	S_3S_1	0.15 → 0			
S_1S_2	0.1	0	S_1S_2	0.1	1	S_1S_2	0.1 → 1			
S_2S_1	0.1	0	S_2S_1	0.1	1	S_2S_1	0.1 → 0			
S_3S_3	0.09	0	S_3S_3	0.09	0	S_3S_3	0.09	1	S_3S_3	0.09 → 1
S_2S_3	0.06	0	S_2S_3	0.06	0	S_2S_3	0.06	1	S_2S_3	0.06 → 0
S_3S_2	0.06	0	S_3S_2	0.06	0	S_3S_2	0.06	0	S_3S_2	0.06 → 1
S_2S_2	0.04	0	S_2S_2	0.04	0	S_2S_2	0.04	0	S_2S_2	0.04 → 0

Symbol	Probabilities	Length	code
S_1S_1	0.25	2	11
S_1S_3	0.15	3	101
S_3S_1	0.15	3	100
S_1S_2	0.1	3	011
S_2S_1	0.1	3	010
S_3S_3	0.09	4	0011
S_2S_3	0.06	4	0010
S_3S_2	0.06	4	0001
S_2S_2	0.04	4	0000

$$H(S^2) = 0.25 \log \frac{1}{0.25} + 2(0.15) \log \frac{1}{0.15} + 2(0.1) \log \frac{1}{0.1}$$

$$+ 0.09 \log \frac{1}{0.09} + 2(0.06) \log \frac{1}{0.06} + 0.04 \log \frac{1}{0.04}$$

$$= 2.971 \text{ bits/MS}$$

$$L = 0.25 \times 2 + (0.15 \times 3)2 + (0.1 \times 3)2 + 0.09 \times 4 + 2(0.06 \times 4)$$

$$+ 0.04 \times 4$$

$$L = 3 \text{ binits/MS}$$

$$\eta_c = \frac{H(S)}{L} = \frac{2.971}{3} = 0.9903 = 99.03\%$$

$$R_{\eta_c} = 1 - 0.9903 = 0.0097 = 00.97\%$$

ii)

$S_1S_1 \quad 0.25 \} \; 2 \qquad S_1S_2 \; 0.25 \rightarrow 2$

$S_1S_3 \quad 0.15 \} \; 2 \qquad S_1S_3 \; 0.15 \rightarrow 1$

$\qquad\qquad\qquad\qquad I \quad 0 \rightarrow 0$

$S_3S_1 \quad 0.15 \; \} \; 1 \qquad S_3S_{\cancel{8}1} \; 0.15 \rightarrow 2$

$S_1S_2 \quad 0.1 \; \} \; 1 \qquad S_1S_2 \; 0.1 \rightarrow 1$

$S_2S_1 \quad 0.1 \; \} \; 1 \qquad S_2S_1 \; 0.1 \rightarrow 0$

$S_3S_3 \quad 0.09 \; \} \; 0 \qquad S_3S_3 \; 0.09 \rightarrow 2$

$S_2S_3 \quad 0.06 \; \} \; 0 \qquad S_2S_3 \; 0.06 \rightarrow 1$

$S_3S_2 \quad 0.06 \; \} \; 0 \qquad S_3S_2 \; 0.06 \} \rightarrow 0 \qquad S_3S_2 \; 0.06 \Rightarrow 2$

$S_2S_2 \quad 0.04 \; \} \; 0 \qquad S_2S_2 \; 0.04 \} \; 0 \qquad S_2S_2 \; 0.04 \Rightarrow 1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad I_1 \quad 0 \rightarrow 0$

Symbol	Probabilities	Length	Code
S_1S_1	0.25	2	22
S_1S_3	0.15	2	21
S_3S_1	0.15	2	12
S_1S_2	0.1	2	11
S_2S_1	0.1	2	10
S_3S_3	0.09	2	02
S_2S_3	0.06	2	01
S_3S_2	0.06	3	002
S_2S_2	0.04	3	001

$H(S^2) = 2.971$ bits / MS

$$L = 2 \times 0.25 + 2(0.15 + 0.15 + 0.1 + 0.1 + 0.09 + 0.06) + 3(0.06 + 0.04)$$

$= 2.1$ trinits / MS

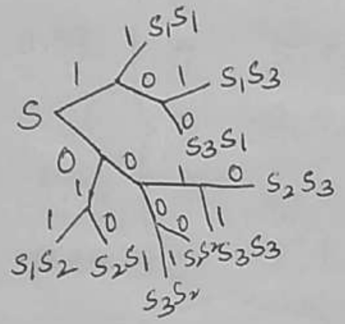
$$H_r(S) = \frac{H(S)}{\log_2 r} = \frac{2.971}{\log_2 3} = 1.874 \text{ bits / MS}$$

2/8

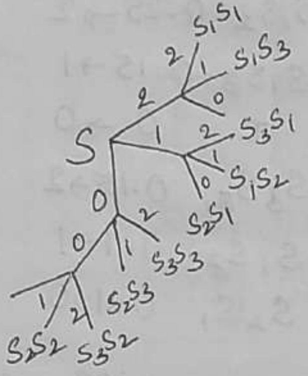
$$\eta_c = \frac{H_r(S)}{L} = \frac{1.874}{2.1} = 0.8924 = 89.24\%$$

$$R_{\eta_c} = 1 - 0.8924 = 0.1076 = 10.76\%$$

Code tree for i)



Code tree for ii)



5/9

Huffman minimum redundancy code : compact code

Steps

① The source symbols are listed in non-increasing order of probabilities. ② Consider the equation

$q = r + (r-1)\alpha$ where q is number of source symbol

r is number of different symbols used in code alphabet.

quantity α is calculated and it should be an integer. If it is not an integer, then dummy symbols with zero probabilities are added to q to get the quantity α an integer. For binary codes α will always be an integer and hence this step is not needed while constructing Huffman binary codes.

③ The last 'r' symbols of source S are now combined into a single composite ~~sign~~ symbol by adding their probabilities to get a reduced source 'S_a'. The symbols of this reduced source are now arranged in the order of non increasing probabilities.

④ The last 'r' symbols of source S_a are now combined to form another composite signal by adding their probabilities to get further reduced source S_b. The symbols of S_b are arranged in the order of non-increasing probabilities.

⑤ This process of combining last 'r' symbols everytime for a new reduced source is continued till we arrive at the last source having 'r' symbols.

⑥ The last source with 'r' message symbols are now encoded with 'r' different code symbols 0, 1, 2 ... r-1.

⑦ In binary coding the last source with 2 symbols are encoded with 0 and 1. As we pass on to the source backwards with 3 symbols, either 0 may be recomposed as 00 and 01 or 1 may be recomposed as 10 and 11 depending on which 2 out of 3 have been combined to get the last reduced source symbol

⑧ As we pass on from source to source backwards, the recomposition of 1 code words each time is done in order to form new code words.

⑨ This procedure is continued till we assign code words to all the source symbols of source S. If any dummy symbols are used, they are discarded.

Problems

① Given the messages S_1, S_2, S_3 and S_4 with respective probabilities of 0.4, 0.3, 0.2 and 0.1, construct a binary code by applying Huffman encoding procedure. Determine efficiency and redundancy of the code so formed.

Solⁿ :- Write the symbols with decreasing order of probabilities.

Since we are finding binary code step ② is not required.

5/9

Source symbols	P_i	code	Source S_a		Source S_b	
			P_i	code	P_i	code
S_1	0.4	1	0.4	1	0.6	0
S_2	0.3	00	0.3	00	0.4	1
S_3	0.2	010	0.3	01		
S_4	0.1	011				

(binary)
↑
$r = 2$
so add last 2 pro till you get r (symbol) probabilities

When composite symbol & source symbol have same probability
Placing the composite ~~single~~ sigmbol as low as possible
when $0.2 + 0.1 = 0.3$ is placed below.

Placing the composite symbol as high as possible
when $0.2 + 0.1 = 0.3$ is placed above S_3 (0.3).

Always show arrow mark when probabilities are shifted.

Symbol	Code	Code	Length	Probabilities
S_1	1~~1~~0	1	1	0.4
S_2	0~~00~~01	00	2	0.3
S_3	0~~10~~01	010	3	0.2
S_4	~~0~~11	011	3	0.1

$$H(S) = 0.4 \log \frac{1}{0.4} + 0.3 \log \frac{1}{0.3} + 0.2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1}$$

$$H(S) = 1.846 \text{ bits } / \text{MS}$$

$$L = 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.2 + 3 \times 0.1 = 1.9 \text{ lr}$$

$$\eta_c = \frac{H(S)}{L} = \frac{1.846}{1.9} = 0.971 = 97.1\%$$

$$R_\eta = 1 - \eta_c = 1 - 0.971 = 0.029 = 2.9\%$$

5|9

Source symbols	P_i	code	Source S_a		Source S_b		Length
			P_i	code	P_i	code	
S_1	0.4	1	0.4	1	0.6	0	1
S_2	0.3	01	0.3	00	0.4	1	2
S_3	0.2	000	0.3	01			3
S_4	0.1	001					3

Placing the composite symbol as high as possible

$H(S) = 1.846$ bits/MS

$L = 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.2 + 3 \times 0.1 = 1.9$

$\eta_c = \frac{H(S)}{L} = 97.1\%$ $\quad R_{\eta_c} = 2.9\%$

---

② Given the messages x_1, x_2, x_3, x_4, x_5 and x_6 with respective probabilities 0.4, 0.2, 0.2, 0.1, 0.07, 0.03 construct binary code by applying Huffman encoding procedure by placing the composite symbol as high as possible. Determine the efficiency and redundancy and write code tree. Repeat the problem by placing the composite symbol as low as possible.

Solⁿ :-

Source symbol	P_i	So Code	Source S_a		Source S_b		Source S_c		Source S_d	
			P_i	code	P_i	code	P_i	code	P_i	code
x_1	0.4	00	0.4	00	0.4	00	0.4	1	0.6	0
x_2	0.2	10	0.2	10	0.2	01	0.4	00	0.4	1
x_3	0.2	11	0.2	11	0.2	10	0.2	01		
x_4	0.1	011	0.1	010	0.2	11				
x_5	0.07	0100	0.1	011						
x_6	0.03	0101								

$$H(S) = 0.4 \log \frac{1}{0.4} + (0.2)\, 2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1} + 0.07 \log \frac{1}{0.07}$$

$$+ 0.03 \log \frac{1}{0.03} = 2.21 \text{ bits/MS}$$

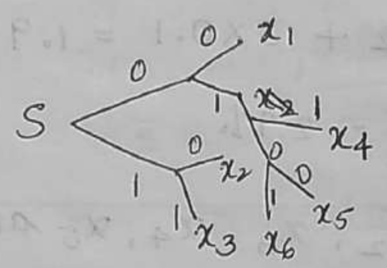
$L = 2\times0.4 + 2\times0.2 + 2\times0.2 + 3\times0.1 + 4\times0.07 + 4\times0.03$

$= 2.3$ binits / MS

$\eta_c = \frac{H(S)}{L} = \frac{2.21}{2.3} = 96.08\%$

$R_{\eta_c} = 1 - 0.9608 = 3.92\%$

Code tree



Placing the composite ~~sig~~ symbol as low as possible

L	Source symbol	P_i	Code	Source S_a		Source S_b		Source S_c		Source S_d	
				P_i	code	P_i	code	P_i	code	P_i	code
1	x_1	0.4	1	0.4	1	0.4	1	0.4	1	0.6	0
2	x_2	0.2	01	0.2	01	0.2	01	0.4	00	0.4	1
3	x_3	0.2	000	0.2	000	0.2	000	0.2	01		
4	x_4	0.1	0010	0.1	0010	0.2	001				
5	x_5	0.07	00100	0.1	00101						
5	x_6	0.03	00101								

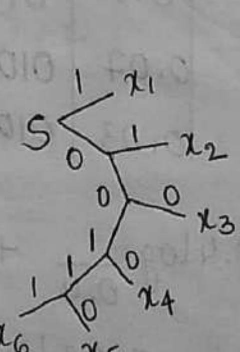
$H(S) = 2.21$ bits / MS

$L = 1\times0.4 + 2\times0.2 + 3(0.2) + 4(0.1) + 5(0.07) + 5(0.03)$

$= 2.3$ binits / MS.

$\eta_c = \frac{H(S)}{L} = \frac{2.21}{2.3} = 96.08\%$

$R_{\eta_c} = 1 - 0.9608 = 3.92\%$



6|9|19

③ Consider a zero memory source with $S = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$ $P = \{0.4, 0.2, 0.1, 0.1, 0.1, 0.05, 0.05\}$

i) Construct a binary Huffman code by placing the composite ~~signal~~ symbol as low as you can.

ii) Repeat the problem by moving the composite symbol as high as possible. iii) In each of these cases compute the variances of word lengths and comment on ~~the~~ the result.

iv) Compute efficiency and redundancy of code and write the code tree.

Sol^n:-

i)

	Source symbol			Source Sa		Source Sb		Source Sd		Src Sd		Src Se	
l_i		P_i	Code	P_i	code	P_i	code	P_i	code	P_i	code	P_i	code
1	S_1	0.4	1	0.4	1	0.4	1	0.4	1	0.4	1	0.6	0
2	S_2	0.2	01	0.2	01	0.2	01	0.2	01	0.4	00	0.4	1
4	S_3	0.1	0010	0.1	0010	0.2	000	0.2	000	0.2	01		
4	S_4	0.1	0011	0.1	0011	0.1	0010	0.2	001				
4	S_5	0.1	0000	0.1	0000	0.1	0011						
5	S_6	0.05	00010	0.1	0001								
5	S_7	0.05	00011										

ii)

	Source Symbol			Source Sa		Source Sb		Source Sc		Source Sd		Source Se	
l_i		P_i	Code	P_i	code	P_i	code	P_i	code	P_i	code	P_i	code
2	S_1	0.4	00	0.4	00	0.4	00	0.4	00	0.4	1	0.6	0
2	S_2	0.2	11	0.2	11	0.2	10	0.2	01	0.4	00	0.4	1
3	S_3	0.1	011	0.1	010	0.2	11	0.2	10	0.2	01		
3	S_4	0.1	100	0.1	011	0.1	010	0.2	11				
3	S_5	0.1	101	0.1	100	0.1	011						
4	S_6	0.05	0100	0.1	101								
4	S_7	0.05	0101										

iv) $H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i}$ **Low**

$= 0.4 \log \frac{1}{0.4} + 0.2 \log \frac{1}{0.2} + 3(0.1) \log \frac{1}{0.1} + 2(0.05) \log \frac{1}{0.05}$

$H(S) = 2.422$ bits /MS

$L = 0.4 \times 1 + 0.2 \times 2 + 0.1 \times 4 + 0.1 \times 4 + 0.1 \times 4 + 0.05 \times 5 + 0.05 \times 5$

$L = 2.5$ binits /MS

$\eta_c = \frac{H(S)}{L} = \frac{2.422}{2.5} = 96.88\%$

$R_{\eta_c} = 1 - \eta_c = 1 - 0.9688 = 0.0312 = 3.12\%$

High

$H(S) = 2.422$ bits /MS

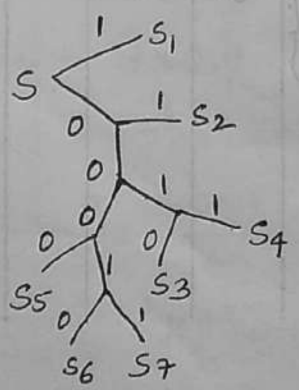
$L = 0.4 \times 2 + 0.2 \times 2 + 0.1 \times 3 + 0.1 \times 3 + 0.1 \times 3 + 0.05 \times 4 + 0.05 \times 4$

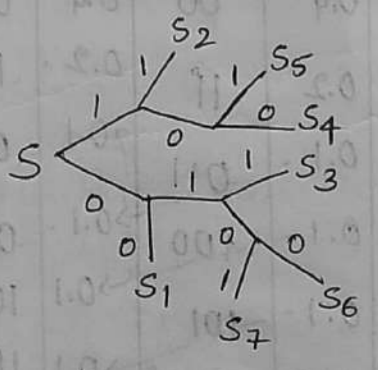
$L = 2.5$ binits /MS

$\eta_c = \frac{H(S)}{L} = \frac{2.422}{2.5} = 96.88\%$

$R_{\eta_c} = 1 - \eta_c = 3.12\%$

Code tree





iii) $Var(x) = E\left[(x-\mu)^2\right]$

$Var(li) = E\left[(li-L)^2\right]$

$= \sum_{i=1}^{7} P_i(li-L)^2$

Low

$Var(li) = 0.4(1-2.5)^2 + 0.2(2-2.5)^2 + 0.1(4-2.5)^2(3)$

$+ 0.05(5-2.5)^2(2)$

$=$ ~~12.2~~ 2.25

High

$Var(li) = 0.4(2-2.5)^2 + 0.2(2-2.5)^2 + 0.1 \times 3(3-2.5)^2$

$+ (0.05 \times 2)(4-2.5)^2$

$= 0.45$

---

④ Consider the source with 8 alphabet A to H with respective probabilities of 0.22, 0.20, 0.18, 0.15, 0.10, 0.08, 0.05, 0.02

i) Construct a binary compact code. Determine the efficiency. Write the code tree.

ii) Construct ternary compact code and determine the efficiency. Write the code tree.

iii) Construct quaternary compact code and determine the code efficiency. Write the code tree.

Compare and contrast on the result.

Soln :- i)	Source Sym	P_i	Code	Source Sa		Source Sb		Source Sc		Source Sd		Source Se		Source Sf	
				P_i	Code	P_i	code	P_i	code	P_i	code	P_i	Code	P_i	c
2	A	0.22	10	0.22	10	0.22	10	0.25	01	0.33	00	0.42	1	0.58	0
2	B	0.20	11	0.20	11	0.20	11	0.22	10	0.25	01	0.33	00	0.42	1
3	C	0.18	000	0.18	000	0.18	000	0.20	11	0.22	10	0.25	01		
3	D	0.15	001	0.15	001	0.15	001	0.18	000	0.20	11				
3	E	0.10	011	0.10	011	0.15	010	0.15	001						
4	F	0.08	0100	0.08	0100	0.10	011								
5	G	0.05	01010	0.07	0101										
5	H	0.02	01011												

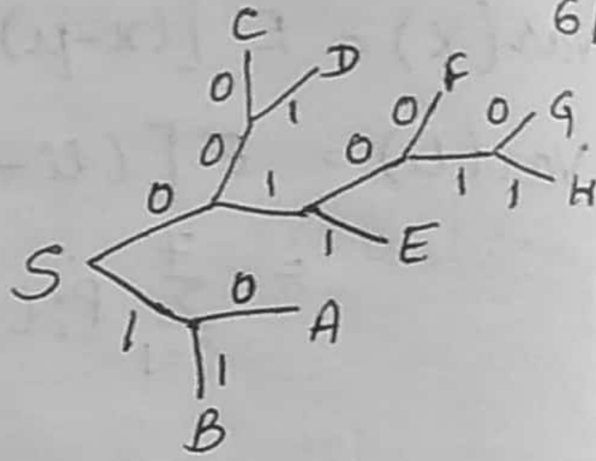
6/9

$H(S) = 2.753$ bits /ms

$L = 2.8$ binits /ms

$\eta_c = \frac{H(S)}{L} = \frac{2.753}{2.8} = 98.32\%$

$R_{\eta_c} = 1 - \eta_c = 1.68\%$



ii) $q = r + (r-1)\alpha$ $r = 3$ $q = 8$

$8 = 3 + 2\alpha$

$5 = 2\alpha$ $\alpha = \frac{5}{2}$ $\Rightarrow$ α should be integer $\therefore$ don't take q as 8

$\alpha = \frac{q-3}{2}$

$\therefore$

$q = r + (r-1)\alpha$

$q = 3 + 2\alpha$

$\alpha = \frac{q-3}{2}$ $\alpha = \frac{9-3}{2} = 3$

$q \Rightarrow 5, 7, 9, 11 \ldots$ to get α as integer

take nxt high no. ~~the~~ of given q

	P_i	Code	Source Sa P_i	Source Sa code	Source Sb P_i	Source Sb code	Source Sb P_i	Source Sb code	L
A	0.22	2	0.22	2	0.25	1	0.53	0	1
B	0.20	00	0.20	00	0.22	2	0.25	1	2
C	0.18	01	0.18	01	0.20	00	0.22	2	2
D	0.15	02	0.15	02	0.18	01			2
E	0.10	10	0.10	10	0.15	02			2
F	0.08	11	0.08	11					2
G	0.05	120	0.07	12					3
H	0.02	121							3
I	0.00	122							3

dummy variable

$H(S) = 2.753$ bits/ms

$L = 0.22 \times 1 + 0.20 \times 2 + 0.18 \times 2 + 0.15 \times 2 + 0.10 \times 2 + 0.08 \times 2$
$+ 0.05 \times 3 + 0.02 \times 3 + 0 \times 3.$

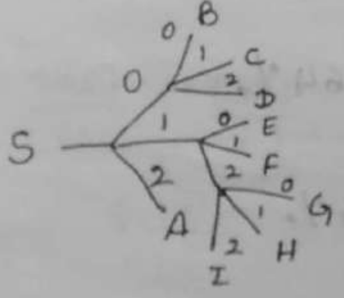
$= 1.85$ trinits / ms

$$H_r(S) = \frac{H(S)}{\log_2 r} = \frac{2.753}{\log_2 3} = 1.737 \text{ bits/ms}$$

$$\eta_c = \frac{H_r(S)}{L} = \frac{1.737}{1.85} = 93.9\%.$$

$$R_{\eta_c} = 1 - \eta_c = 1 - 0.939 = 6.1\%.$$

ii)



iii) $q = r + (r-1)\alpha$ $\quad r\cancel{\alpha} = 4 \quad q = 8$

$8 = 4 + (4-1)\alpha \Rightarrow 8 = 4 + 3\alpha$

$4 = 3\alpha \quad \alpha = \frac{4}{3}$

$q = 4 + 3\alpha$

$\alpha = \frac{q-4}{3}$

$q = 7, 10, 13 \ldots$

9/9/19

$\alpha = \frac{10-4}{3} = \frac{6}{3} = 2$

	P_i	Code	Source Sa		Source Sb		Source Sc	
			P_i	code	P_i	code	P_i	code
A	0.22	1	0.22	1	0.22		0.4	0
B	0.20	2	0.20	2	0.20		0.22	1
C	0.18	3	0.18	3	0.18		0.20	2
D	0.15	00	0.15	00	0.15		0.18	3
E	0.10	01	0.10	01	0.15			
F	0.08	02	0.08	02	0.10			
G	0.05	030	0.07	03				
H	0.02	031	~~0.02~~					
I	0.00	032						
J	0.00	033						

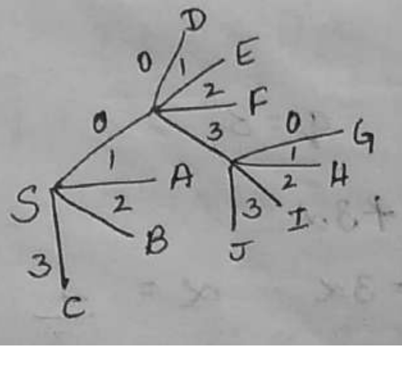
$H(S) = \frac{2.753}{\log_2 4} = 1.3765$ bits / MS

$L = 0.22\times1 + 0.20\times1 + 0.18\times1 + 0.15\times2 + 0.10\times2 + 0.08\times2 + 0.05\times3 + 0.02\times3$

$= 1.47$ quad bits / MS

$\eta_c = \frac{H(S)}{L} = \frac{1.3765}{1.47} = 93.64\%$

$R_{\eta_c} = 1 - 0.9364 = 6.36\%$



⑤ Apply Huffman encoding procedure for the following set of messages and determine the i) efficiency of binary code so formed. 9/9/19

x_1 x_2 x_3
0.7 0.15 0.15

ii) If the same technique is applied to II order extension how much will the efficiency be improved?

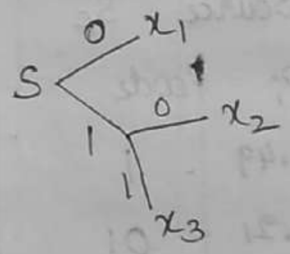
iii) Apply Huffman ternary and quaternary coding for II extended source, compute efficiency & redundancy?

iv) Write the code trees for all the codes.

Solⁿ :-

i)

Source symbol	P_i	Code	Source Sa	
			P_i	code
x_1	0.7	0	0.7	0
x_2	0.15	10	0.3	1
x_3	0.15	11		



$$H(S) = 0.7 \log \frac{1}{0.7} + 2(0.15) \log \frac{1}{0.15}$$

$H(S) = 1.181$ bits / MS

$L = 0.7 \times 1 + 0.15 \times 2 + 0.15 \times 2 = 1.3$ binits / ms

$$\eta_c = \frac{H(S)}{L} = \frac{1.181}{1.3} = 90.8\%.$$

$R_{\eta_c} = 9.2\%.$

ii)

$x_1 x_2 = 0.105$ $x_1 x_1 = 0.49$
$x_1 x_3 = 0.105$ $x_3 x_1 = 0.105$
$x_2 x_1 = 0.105$ $x_3 x_2 = 0.0225$
$x_2 x_2 = 0.0225$ $x_3 x_3 = 0.0225$
$x_2 x_3 = 0.0225$

Soure symbols	P_i	code	Source Sa		Source S_b		Source S_c		Source S_d	
			P_i	code	P_i	code	P_i	code	P_i	code
$x_1 x_1$	0.49	1	0.49	1	0.49	1	0.49	1	0.49	1
$x_1 x_2$	0.105	001	0.105	001	0.105	001	0.105	001	0.195	000
$x_1 x_3$	0.105	010	0.105	010	0.105	010	0.105	010	0.105	001
$x_2 x_1$	0.105	011	0.105	011	0.105	011	0.105	011	0.105	010
$x_3 x_1$	0.105	0000	0.105	0000	0.105	0000	0.105	0000	0.105	011
$x_2 x_2$	0.0225	000110	0.045	00010	0.045	00010	0.09	0001		
$x_2 x_3$	0.0225	000111	0.0225	000110	0.045	00011				
$x_3 x_2$	0.0225	000100	0.0225	000111						
$x_3 x_3$	0.0225	000101								

Source S_e		Soure S_f		Source S_g	
P_i	code	P_i	code	P_i	code
0.49	1	0.49	1	0.51	0
0.21	01	0.3	00	0.49	1
0.195	000	0.21	01		
0.105	001				

$H^2(S) = 2H(S) = 2(1.181) = 2.362$ bits/ms

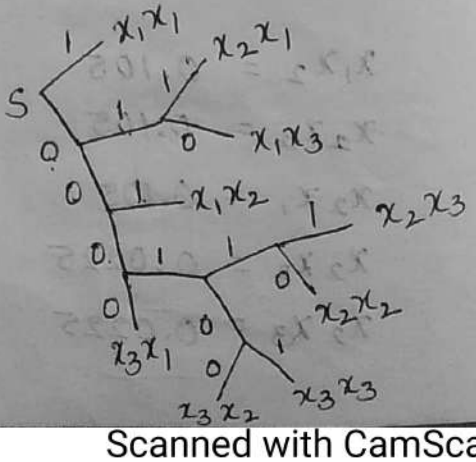
$L = 0.49 + 3(0.105 + 0.105 + 0.105) + 4(0.105) + 6(0.0225 +$

$0.0225 + 0.0225 + 0.0225)$

$= 2.395$ binits/ms

$\eta_c = \frac{H^2(S)}{L} = 98.624\%$

$R_{\eta_c} =$ ~~0.76%~~ 1.38%.



9/9

iii) $q = r + (r-1)\alpha$ $\quad q = 9 \quad r = 3 \quad$ ternary

$q = 3 + 2\alpha$

$\alpha = \frac{q-3}{2}$ $\qquad \alpha = \frac{9-3}{2} = 3.$

$q = 5, 7, 9, 11 \ldots \qquad q = 9.$

Source symbols	P_i	Code	Source S_a		S_b		S_c	
			P_i	code	P_i	code	P_i	code
x_1x_1	0.49	0	0.49	0	0.49	0	0.49	0
x_1x_2	0.105	10	0.105	10	0.195	2	0.315	1
x_1x_3	0.105	11	0.105	11	0.105	10	0.195	2
x_2x_1	0.105	12	0.105	12	0.105	11		
x_3x_1	0.105	20	0.105	20	0.105	12		
x_2x_2	0.0225	22	0.0675	21				
x_2x_3	0.0225	210	0.0225	22				
x_3x_2	0.0225	211						
x_3x_3	0.0225	212						

$H^2(S) = 2.362$ bits/MS

$L = 0.49 \times 1 + 0.105\times2 + 0.105 \times 2 + 0.105\times2 + 0.105\times2$

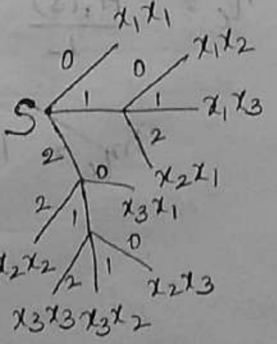
$+ 0.0225 \times 2 + (0.0225\times3)3$

$L = 1.5775$

$H_r(S) = \frac{H^2(S)}{\log_2 r} = \frac{2.362}{\log_2 3} = 1.49$ bits/MS

$\eta_c = \frac{H_r(S)}{L} = \frac{1.49}{1.5775} = 94.45\%$

$R_{\eta_c} = 1 - \eta_c = 5.55\%$



Quaternary coding

9/9

$q = r + (r-1)\alpha$

$\alpha = \frac{q-r}{r-1}$ $\alpha = \frac{q-4}{3}$ $\alpha = \frac{10-4}{3} = 2$

$q = 7, 10, 13 \ldots$ $q = 10$

Source symbol	P_i	Code	Sa		Sb	
			P_i	code	P_i	code
$x_1 x_1$	0.49	0	0.49	0	0.49	0
$x_1 x_2$	0.105	2	0.105	2	0.3	1
$x_1 x_3$	0.105	3	0.105	3	0.105	2
$x_2 x_1$	0.105	10	0.105	10	0.105	3
$x_3 x_1$	0.105	11	0.105	11		
$x_2 x_2$	0.0225	13	0.0675	12		
$x_2 x_3$	0.0225	120	0.0225	13		
$x_3 x_2$	0.0225	121				
$x_3 x_3$	0.0225	122				
I	0	123				

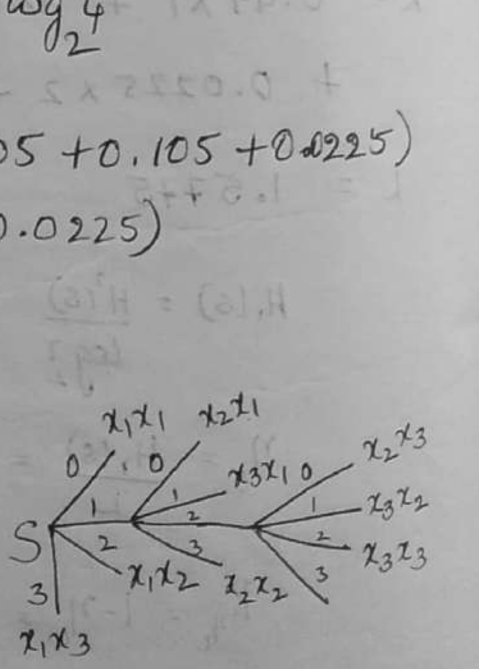
$H^2(S) = 2.362$ $H_r(S) = \frac{H^2(S)}{\log_2 4} = \frac{2.362}{\log_2 4} = 1.181$ bits/ms

$L = 0.49 \times 1 + 0.105 + 0.105 + 2(0.105 + 0.105 + 0.0225)$
$+ 3(0.0225 + 0.0225 + 0.0225)$

$L = 1.3675$

$\eta_c = \frac{H_r(S)}{L} = \frac{1.181}{1.3675} = 86.36\%$

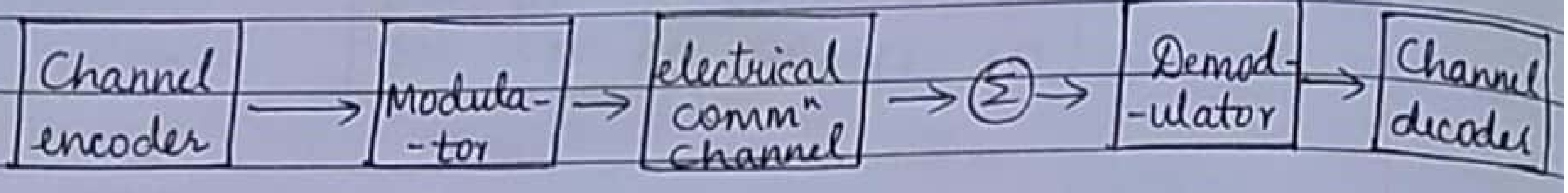
$R_{\eta_c} = 1 - \eta_c = 13.64\%$



11/11/19

Module -~~5~~3

Information Channels



Representation of a channel

Source $A = \{a_1, a_2 \dots a_r\}$ r no. of symbols at i/p

At output $B = \{b_1, b_2, \dots b_s\}$ s no. of symbols at o/p

Conditional probability is $P(b_j/a_i)$

Probability of receiving the symbol b_j by transmitting the symbol a_i.

$$A\left\{\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array}\right. \rightarrow \boxed{P(b_j/a_i)} \rightarrow \left.\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_s \end{array}\right\} B$$

We can have $r \times s$ no. of conditional probabilities

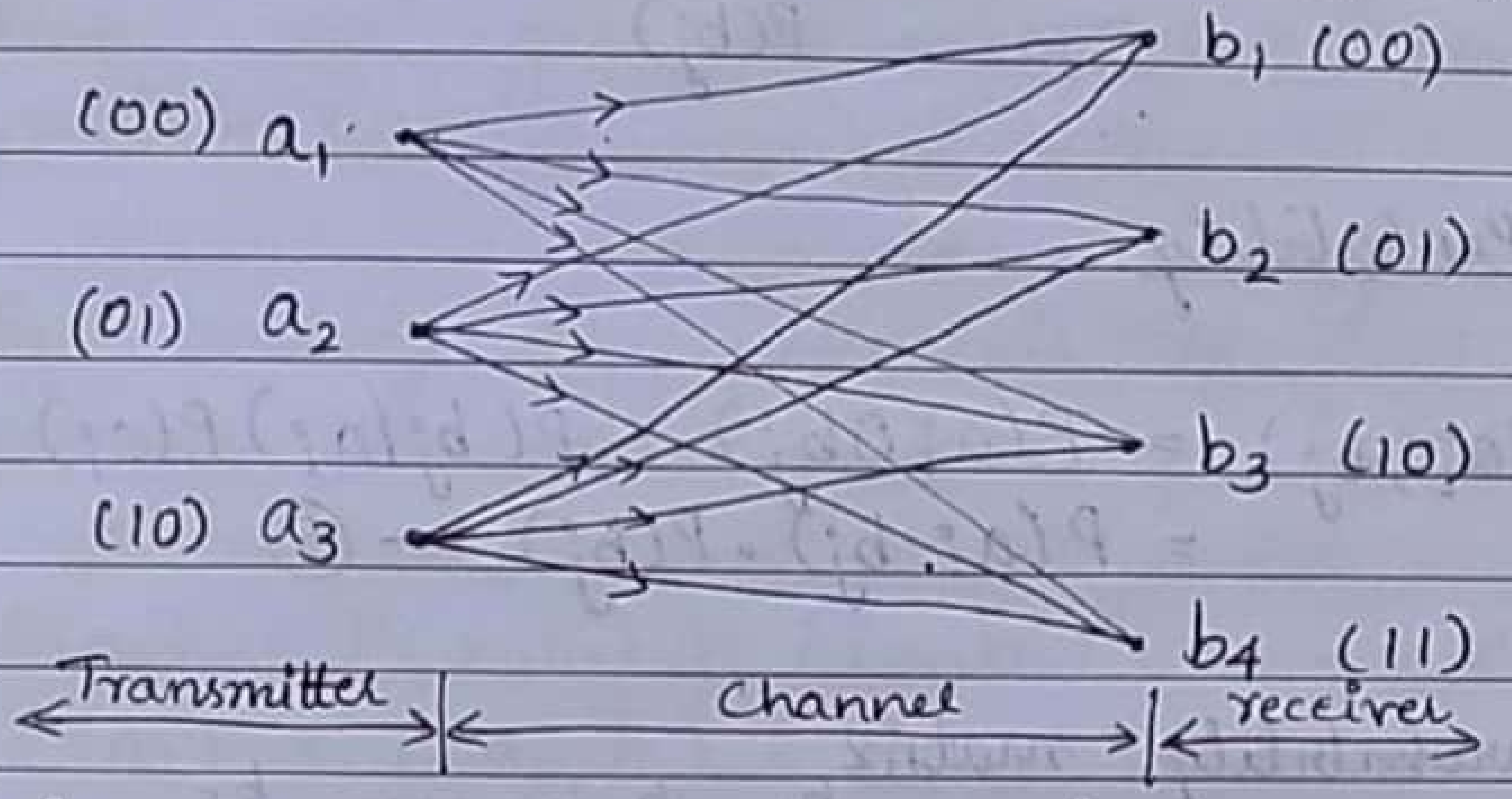
Channel matrix or Noise matrix is represented by

$$P(b_j/a_i) \text{ or } P(B/A) = \begin{array}{c|cccc|} & b_1 & b_2 & \cdots & b_s \\ a_1 & P(b_1/a_1) & P(b_2/a_1) & \cdots & P(b_s/a_1) \\ a_2 & P(b_1/a_2) & P(b_2/a_2) & \cdots & P(b_s/a_2) \\ a_3 & \vdots & & & \\ \vdots & \vdots & & & \\ a_r & P(b_1/a_r) & P(b_2/a_1) & \cdots & P(b_s/a_r) \end{array}$$

11/11/19

Representation of channel Diagram / Noise Diagram

$A = \{a_1, a_2, a_3\}$ $B = \{b_1, b_2, b_3, b_4\}$



$P(b_1|a_1) = P_{11}$

$P(b_2|a_1) = P_{12}$

$P(b_3|a_1) = P_{13}$

$P(b_4|a_1) = P_{14}$

If a_1 has error in 1^{st} place it may be received as 10, in 2^{nd} place → 01, both place 11, no error oc

$P_{11} + P_{12} + P_{13} + P_{14} = 1$

In general

$$P_{11} + P_{12} + P_{13} + ---- + P_{1S} = 1$$

$$\sum_{j=1}^{s} P(b_j/a_1) = 1 \quad — ①$$

$$P(a_1) + P(a_2) + --- + P(a_r) = 1$$

The sum of probability of occurrence of all the symbols at the input side is 1

$$\sum_{i=1}^{r} \{ P(a_i) = 1 \quad — ②$$

$$P(b_1) = P(b_1/a_1) + P(b_2/a_2) + ... + P(b_1/a_r)$$

$$P(b_2) = P(b_2/a_1) + P(b_2/a_2) + ... + P(b_2/a_r)$$

$$P(b_s) = P(b_s/a_1) + P(b_s/a_2) + \dots + P(b_s/a_r) \quad — ③$$

$$P(a_i/b_j) = \frac{P(b_j/a_i)\,P(a_i)}{P(b_j)} \quad — ④ \Rightarrow \text{Baye's Theorem}$$

Joint Probability

$$P(a_i, b_j) = P(a_i \cap b_j) = P(b_j/a_i)\,P(a_i)$$
$$= P(a_i . b_j) . P(b_j) \quad — ⑤$$

Joint probability matrix

$$P(b_j/a_i)\,P(a_i) = \begin{array}{c|ccc|} & b_1 & b_2 \dots\dots & b_s \\ a_1 & P(b_1/a_1)P(a_1) & \cdots & P(b_s/a_1)P(a_1) \\ a_2 & P(b_1/a_2)P(a_2) & \cdots & P(b_s/a_2)P(a_2) \\ a_3 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_r & P(b_1/a_r)P(a_r) & \cdots & P(b_s/a_r)P(a_r) \end{array}$$

$$P(a_i, b_j) \text{ or } P(A,B) = \begin{array}{c|cccc|} & b_1 & b_2 & & b_s \\ a_1 & P(a_1,b_1) & P(a_1,b_2) & \cdots & P(a_1,b_s) \\ a_2 & P(a_2,b_1) & P(a_2,b_2) & \cdots & P(a_2,b_s) \\ \vdots & \vdots & \vdots & & \vdots \\ a_r & P(a_r,b_1) & P(a_r,b_2) & \cdots & P(a_r,b_s) \end{array}$$

⇓

Joint Probability matrix
(JPM)

3 properties of Joint Probability Matrix

By adding all the elements of I column we get the probability of I o/p symbol b_1

$$P(b_1) = P(a_1,b_1) + P(a_2,b_1) + \dots + P(a_r,b_1)$$

$P(b_s) = P(a_1, b_s) + P(a_2, b_s) + \cdots + P(a_r, b_s)$

By adding the elements (probabilities) of JPM column wise ~~then~~ we get the respective output symbol probabilities

2. By adding the elements of JPM row wise, we obtain the probabilities of input symbols.

$P(a_1) = P(a_1, b_1) + P(a_1, b_2) + \cdots + P(a_1, b_s)$
$P(a_2) = P(a_2, b_1) + P(a_2, b_2) + \cdots + P(a_2, b_s)$
$\vdots$
$P(a_r) = P(a_r, b_1) + P(a_r, b_2) + \cdots + P(a_r, b_s)$

3. The sum of all the elements of JPM is unity.

Problems

① In a communication system, transmitter has 3 input symbols $A = \{a_1, a_2, a_3\}$ and receiver also has 3 output symbols $B = \{b_1, b_2, b_3\}$. The matrix given below is a JPM with some marginal probabilities.

	b_1	b_2	b_3
a_1	$1/12$	*	$5/36$
a_2	$5/36$	$1/9$	$5/36$
a_3	*	$1/6$	*
$P(b_j)$	$\frac{1}{3}$	$\frac{14}{36}$	*

i) Find missing probabilities
ii) Find $P(b_3/a_1)$ and $P(a_1/b_3)$

11/11/19

iii) Are the events a_1 and b_1 statistically independent? Why?

Sol i) $P(b_1) = \frac{1}{12} + \frac{5}{36} + *$

$$\frac{1}{3} = \frac{1}{12} + \frac{5}{36} + *$$

$$* = \frac{1}{9} = P(b_1/a_3)$$

$$P(b_2) = * + \frac{1}{9} + \frac{1}{6}$$

$$\frac{14}{36} = * + \frac{1}{9} + \frac{1}{6}$$

$$* = \frac{1}{9} = P(b_2/a_1)$$

$P(b_j/a_i) = 1$

$$1 = \frac{1}{12} + \frac{5}{36} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{6} + \frac{5}{36} + \frac{5}{36} + *$$

$$* = 0 \quad P(b_3/a_3)$$

$$P(b_3) = \frac{5}{36} + \frac{5}{36} + 0 = \frac{5}{18}$$

ii) $P(b_3/a_1) = \frac{5}{36}$ $\qquad P(a_1/b_3)$

iii) $P(a_1 \cap b_1) = P(a_1) . P(b_1)$

$$1/12 \neq \frac{1}{3} . \frac{1}{3}$$

a_1 and b_1 are not statistically independent.

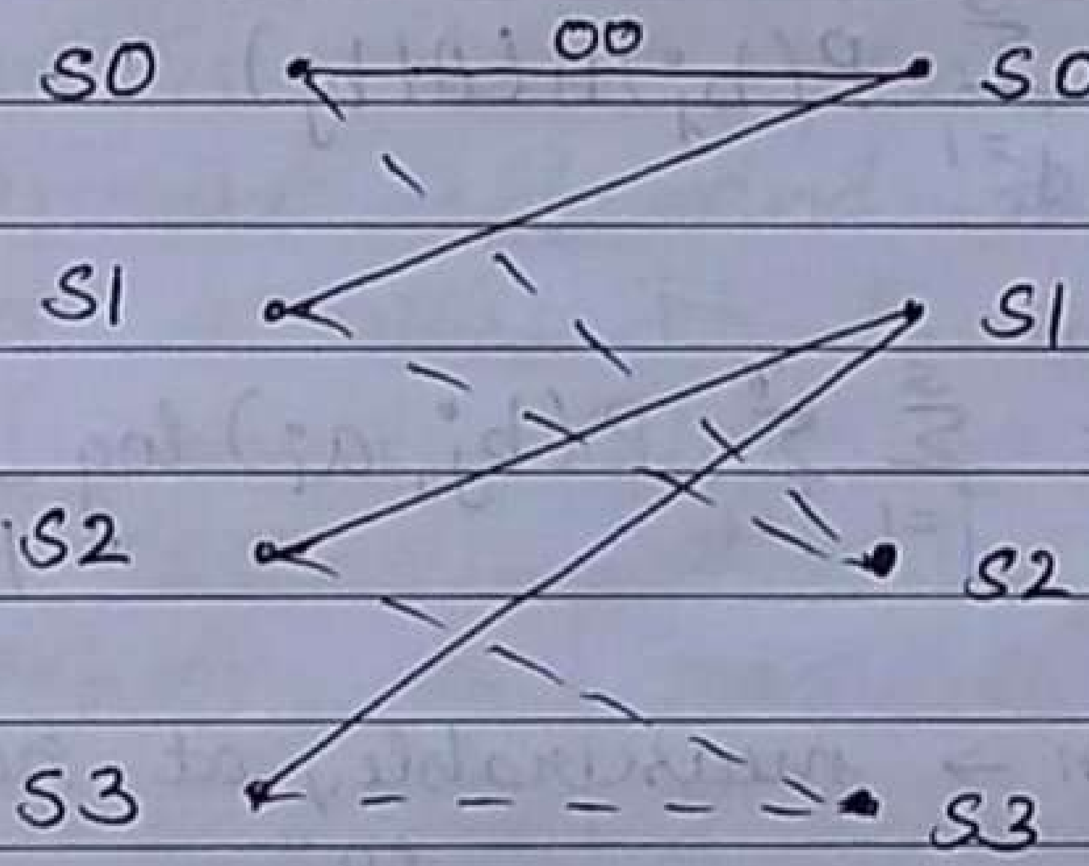
Code Trellis

Present state	Binary description	Input	Next state	Binary description	d_l	d_{l-1}	d_{l-2}	O/p	
								$c^{(1)}$	$c^{(2)}$
S_0	00	0	S0	00	0	0	0	0	0
		1	S2	10	1	0	0	1	1
S_1	01	0	S0	00	0	0	1	1	1
		1	S2	10	1	0	1	0	0
S_2	10	0	S1	01	0	~~0~~1	0	1	0
		1	S3	11	1	1	0		
S_3	~~00~~11	0	S1	01	0	1	1		
		1	S3	11	1	1	1		

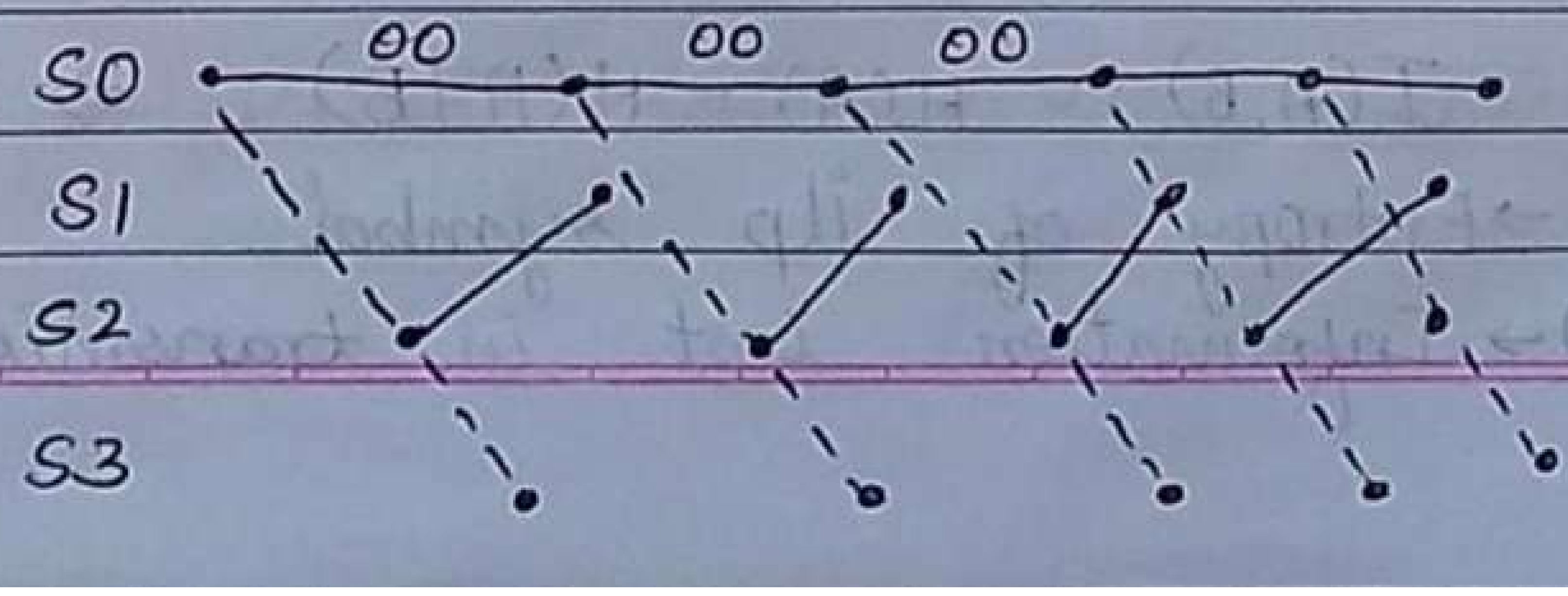
dotted lines → input → ~~0~~1
dark lines → input → 0

Code Trellis

10111
{11, 10, 00, 01, 10, 01, 11}



Trellis Diagram



1|19

Entropy functions and Equivocation

1. Priori entropy :- Entropy of i/p symbols before transmitting

$$H(A) = \sum_{i=1}^{r} P(a_i) \log \frac{1}{P(a_i)} \text{ bits/MS}$$

2. Posteriori ~~Positive~~ entropy (also called as conditional entropy)
Entropy at reception after transmitting i/p symbol

$$H(A/b_j) = \sum_{i=1}^{r} P(a_i/b_j) \log \frac{1}{P(a_i/b_j)} \text{ bits/MS}$$

i varies from 1 to s

Average value of all conditional entropies is called equivocation. Amount of ~~noise~~ info^m. lost during transmission

$$H(A|B) = E[(H(A|b_j)]$$

$$= \sum_{j=1}^{s} P(b_j) H(A|b_j)$$

$$H(B/A) = \sum_{j=1}^{s} \sum_{i=1}^{r} P(b_j, a_i) \log \frac{1}{P(b_j|a_i)}$$

$\uparrow$

equivocation → measurable eq^n at receiver

Mutual Information I(A,B)

$$I(A,B) = H(A) - H(A|B) \Rightarrow \text{Mutual}$$

H(A) → Entropy of i/p symbol

H(A|B) → Information lost in transmission

$I(B,A) = H(B) - H(B/A)$

Properties of mutual information

1. MI of a channel is symmetric

$$I(A,B) = I(B,A)$$

Proof :-

2. MI is always non negative i.e $I(A,B) \geq 0$

Proof :-

3. The MI of a channel may be expressed interms of entropy of the channel output as $I(A,B) = H(B) - H(B/A)$

4. The MI is related to the Joint entropy of the channel by

$$I(A,B) = H(A) + H(B) - H(A,B)$$

$H(B) \rightarrow$ Entropy of output symbol

Problems

① A transmitter has alphabet consisting of 5 letters $\{a_1\ a_2\ a_3\ a_4\ a_5\}$ and the receiver has alphabet of 4 letters $\{b_1\ b_2\ b_3\ b_4\}$ The JP's of the system is given below.

$$P(A,B) = \begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & 0.25 & 0 & 0 & 0 \\ a_2 & 0.10 & 0.30 & 0 & 0 \\ a_3 & 0 & 0.05 & 0.10 & 0 \\ a_4 & 0 & 0 & 0.05 & 0.1 \\ a_5 & 0 & 0 & 0.05 & 0 \end{array}$$

Compute different entropies of this channel.

$H(A) = ?$ $H(B) = ?$ $H(A/B) = ?$ $H(B/A) = ?$

$$H(A/B) = \sum_{j=1}^{s} P(b_j)\, H(A/b_j)$$

$$H(A) = \sum_{i=1}^{r} P(a_i) \log \frac{1}{P(a_i)}$$

$P(a_1) = 0.25$

$P(a_2) = 0.40$

$P(a_3) = 0.15$

$P(a_4) = 0.15$

$P(a_5) = 0.05$

$$H(A) = 0.25 \log \frac{1}{0.25} + 0.4 \log \frac{1}{0.4} + 0.15 \log \frac{1}{0.15} + 0.15 \log \frac{1}{0.15} + 0.05 \log \frac{1}{0.05}$$

$H(A) = 2.066$ bits/ms

$$H(B) = \sum_{j=1}^{s} P(b_j) \log \frac{1}{P(b_j)}$$

$P(b_1) = 0.35$ $P(b_2) = 0.35$

$P(b_3) = 0.2$ $P(b_4) = 0.1$

$$H(B) = 0.35 \log \frac{1}{0.35} + 0.35 \log \frac{1}{0.35} + 0.2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1}$$

$H(B) = 1.8568$ bits/ms ≈ 1.857

~~H(A,B)~~ = $H(A,B) = \sum_{j=1}^{4} \sum_{i=1}^{5} P(a_i, b_j) \log \frac{1}{P(a_i, b_j)}$

$$= 0.25 \log \frac{1}{0.25} + 0.10 \log \frac{1}{0.10} + 0.30 \log \frac{1}{0.3} + 0.05 \log \frac{1}{0.05}$$

$$+ 0.10 \log \frac{1}{0.1} + 0.05 \log \frac{1}{0.05} + 0.1 \log \frac{1}{0.1} + 0.05 \log \frac{1}{0.05}$$

$H(A,B) = 2.666$ bits /MS

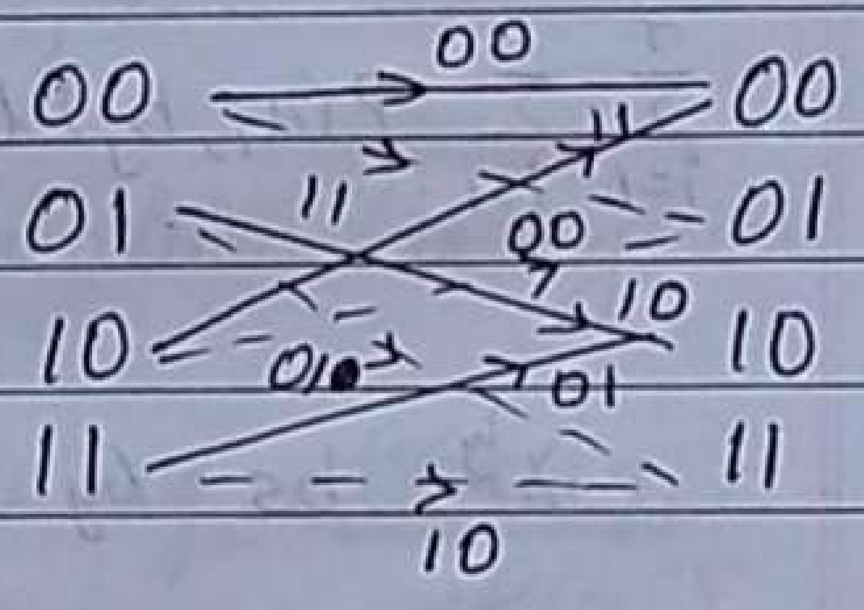
$$I(A, B) = H(A) + H(B) - H(A, B)$$
$$= 2.066 + 1.8568 - 2.666$$
$$= 1.2568 \text{ bits /MS}$$

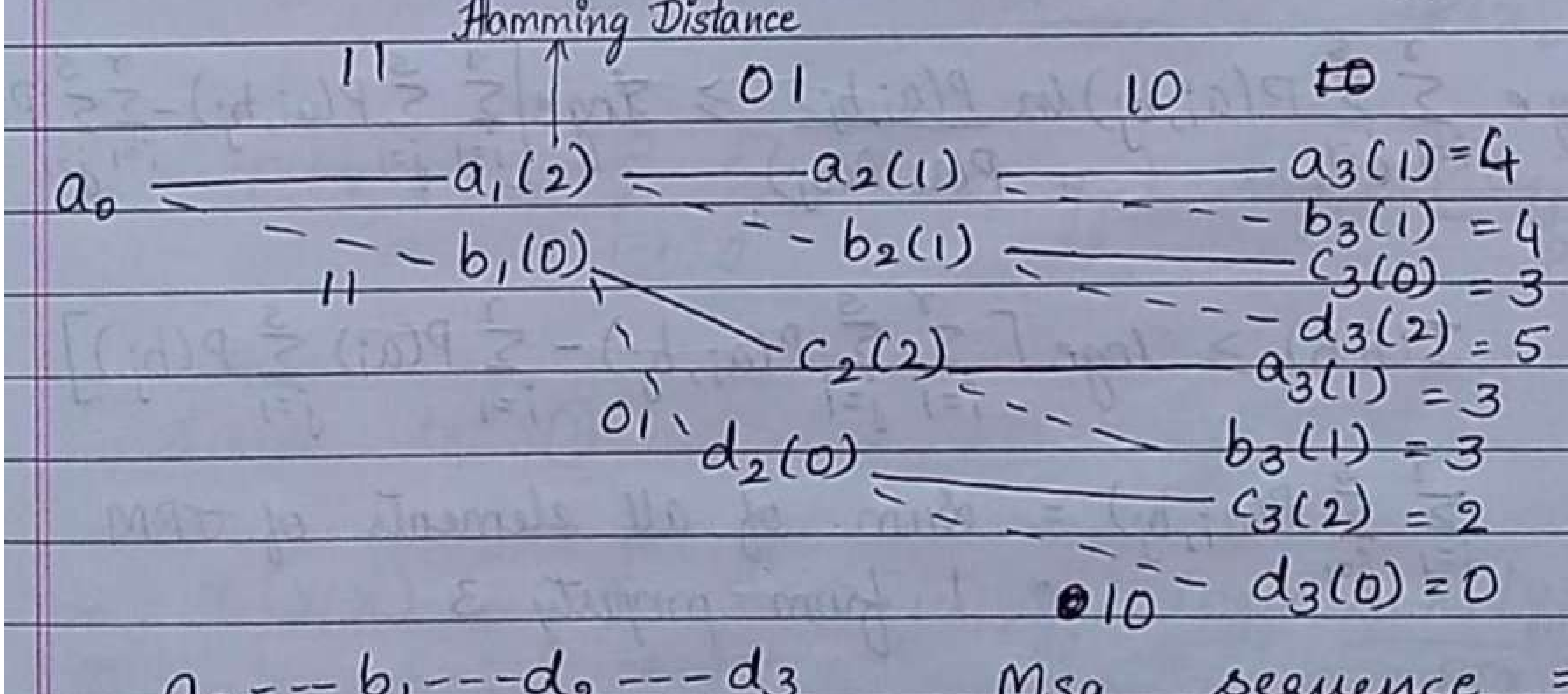
$$H(B/A) = H(A,B) - H(A) = 2.666 - 2.066 = 0.6 \text{ bits/ms}$$
$$H(A/B) = H(A,B) - H(B) = 2.666 - 1.8568 = 0.809 \text{ bits/Ms}$$

$I(A,B) = H(A) - H(A/B)$ or $I(A,B) = H(B) - H(B/A)$

Viterbi Decoding

Input : 110110





a_0 --- b_1 --- d_2 --- d_3 Msg sequence = 111

To calculate error → 110110 ← i/p given

110110 ← o/p got at last

0 → error

Property - 2 : The mutual information is always non-negative

$$I(A,B) \geq 0$$

Proof :-

$$I(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 \frac{P(a_i, b_j)}{P(a_i)P(b_j)}$$

$$= \log e \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \ln \frac{P(a_i, b_j)}{P(a_i)P(b_j)}$$

$$\ln \frac{1}{x} \geq 1 - x$$

$$x = \frac{P(a_i)P(b_j)}{P(a_i, b_j)}$$

$$\ln \frac{P(a_i, b_j)}{P(a_i)P(b_j)} \geq \left[1 - \frac{P(a_i)P(b_j)}{P(a_i, b_j)}\right] \quad -①$$

X^{ly} bs ~~log~~ of eq ① by $P(a_i, b_j)$

$$\sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \ln \frac{P(a_i, b_j)}{P(a_i)P(b_j)} \geq \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \left[1 - \frac{P(a_i)P(b_j)}{P(a_i, b_j)}\right]$$

X^{ly} bs by $\log e$

$$\log e \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \ln \frac{P(a_i, b_j)}{P(a_i)P(b_j)} \geq \log e \left[\sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) - \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i)P(b_j)\right]$$

$$I(A,B) \geq \log e \left[\sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) - \sum_{i=1}^{r} P(a_i) \sum_{j=1}^{s} P(b_j)\right]$$

$\sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j)$ = Sum of all elements of JPM

= 1 from property 3

$$\sum_{i=1}^{r} P(a_i) = 1 \qquad \sum_{i=1}^{s} P(b_j) = 1$$

$$I(A,B) \geq \log_e [1-(1)(1)]$$

$$I(A,B) \geq 0 \rightarrow \text{proved}$$

Property 1 :- Mutual information of the channel is symmetric

$$I(X,Y) = I(Y,X)$$

Proof :- $P(x_i, y_j) = P(x_i|y_j)\, P(y_j)$

$P(x_i, y_j) = P(y_j|x_i)\, P(x_i)$

$P(x_i, y_j)$ is the joint probability that x_i is transmitted and y_j is received.

$$P(x_i|y_j)\, P(y_j) = P(y_j|x_i)\, P(x_i)$$

$$\frac{P(x_i|y_j)}{P(x_i)} = \frac{P(y_j|x_i)}{P(y_j)} \quad \text{— ①}$$

The average mutual information is given by

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i|y_j)}{P(x_i)}$$

$$I(Y;X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(y_j|x_i)}{P(y_j)}$$

from eq ①

$$I(Y;X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i|y_j)}{P(x_i)}$$

$$= I(X;Y)$$

$$I(X;Y) = I(Y;X)$$

Hence proved.

14/11 **Problem**

② A transmitter transmits 5 symbols with probabilities 0.2, 0.3, 0.2, 0.1, 0.2. Given the channel matrix P(B/A). Calculate i) H(B) ii) H(A,B) iii) H(A) iv) I(A,B)

$$P(B/A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Sol.

$P(b_1) = \cancel{1.25}\ 1 \times 0.2 = 0.2$

$P(b_2) = \cancel{1.083}\ 1 \times 0.3 = 0.3$

$P(b_3) = \cancel{1.66}\ 1 \times 0.2 = 0.2$

$P(b_4) = \cancel{0}\ 1 \times 0.1 = 0.1$

$P(b_5) = 1 \times 0.2 = 0.2$

$H(B) =$

$P(b_1) = 1.25$

$P(b_2) = 1.083$

$P(b_3) = 1$

$P(b_4) = 2/3 = 0.66$

$$P(a_i, b_j) = P(b_j / a_i) . P(a_i)$$

Given $P(b_j/a_i)$

$$P(a_i, b_j) = \begin{bmatrix} 0.2\ {}^{1}/_{5} & 0 & 0 & 0 \\ 0.075\ {}^{3}/_{40} & 0.225\ {}^{9}/_{40} & 0 & 0 \\ 0 & 1/15 & 2/15 & 0 \\ 0 & 0 & 1/30 & 1/15 \\ 0 & 0 & 1/5 & 0 \end{bmatrix}$$

14/11

$P(b_1) = $ 0.275 $= 11/40$

$P(b_2) = $ ~~0.3~~ $7/24$

$P(b_3) = $ ~~0.21~~ $11/30$

$P(b_4) = 1/15$

i) $$H(B) = \frac{11}{40}\log\left(\frac{40}{11}\right) + \frac{7}{24}\log\left(\frac{24}{7}\right) + \frac{11}{30}\log\left(\frac{30}{11}\right) + \frac{1}{15}\log 15$$

$H(B) =$ ~~0.5484 bits/MS~~ 1.8218 bits/MS.

ii) $$H(A,B) = \sum_{j=1}^{4}\sum_{i=1}^{5} P(a_i, b_j)\log\frac{1}{P(a_i,b_j)}$$

$$= \frac{1}{5}\log 5 + \frac{3}{40}\log\frac{40}{3} + \frac{9}{40}\log\frac{40}{9} + \frac{1}{15}\log 15 + \frac{2}{15}\log\frac{15}{2}$$
$$+ \frac{1}{30}\log 30 + \frac{1}{15}\log 15 + \frac{1}{5}\log 5$$

$H(A,B) = 2.765$ bits/ms

iii) $H(A)$

$P(a_1) = 1/5$

$P(a_2) = 3/10$

$P(a_3) = 1/5$

$P(a_4) = 1/10$

$P(a_5) = 1/5$

$$H(A) = \frac{1}{5}\log 5 + \frac{3}{10}\log\frac{10}{3} + \frac{1}{5}\log 5 + \frac{1}{10}\log 10 + \frac{1}{5}\log 5$$

$H(A) = 2.246$ bits/ms

iv) $I(A,B) = H(A) - H(A|B)$

$H(A|B) = H(A,B) - H(B) = 2.765 - 1.8218 = 0.9432$ bit/M

$I(A,B) = 2.246 - 0.9432 = 1.3028$ bits/ms

Rate of information Transmission over a discrete channel

$$H(A) = \sum_{i=1}^{r} P(a_i) \log \frac{1}{P(a_i)} \text{ bits/MS}$$

Discritised channel emits r_s message symbol/sec

$$R_{in} = H(A) r_s \text{ bits/sec}$$

Input rate is R_{in}
Transmission rate R_t

$$R_t = [H(A) - H(A|B)] r_s \text{ bits/sec}$$

$$R_t = I(A, B) r_s \text{ bits/sec}$$

$$I(A, B) = I(B, A)$$
$$\therefore R_t = [H(B) - H(B|A)] r_s \text{ bits/sec}$$

Capacity of a channel

It is the maximum transmission rate of a channel.

$$C = \text{Max}\{R_t\}$$
$$C = \text{Max}[H(A) - H(A|B)] \text{ ~~bits~~ } r_s$$

Shannon's capacity of a channel is defined in 2 methods : Shannon's II theorem

i) Positive statements : $R_t \leq C$
ii) Negative statement : $R_t > C$

14/11

Channel efficiency and redundancy

Channel efficiency $\eta_{ch} = \frac{R_t}{C} \times 100\%$.

$$= \frac{[H(A) - H(A|B)]r_s}{Max[H(A) - H(A|B)]r_s} \times 100\%.$$

$$\eta_{ch} = \frac{I(A,B)}{Max[I(A,B)]} \times 100\%.$$

Redundancy $R_{\eta_{ch}} = 1 - \eta_{ch}$.

Special Channels

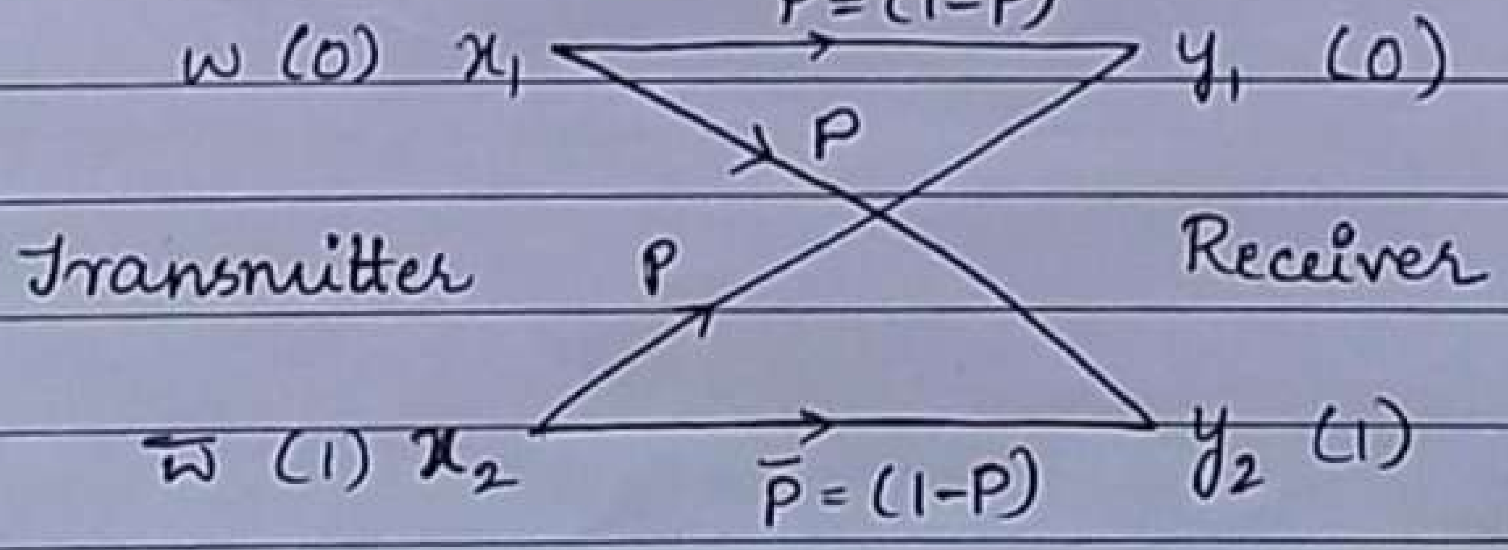
Symmetric channels (not in syllabus)

Elements of I row is arranged in subsequent rows in different order.

Eg :- $\begin{bmatrix} 1/3 & 2/3 & 0 \\ 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \end{bmatrix}$

Binary symmetric channels

It has only 2 inputs and 2 outputs.



x_1 transmitted → 0 if we receive 0 from (x_1 or x_2) then y_1

x_2 transmitted → 1 — " — " — 1 from (x_1 or x_2) then y_2

14/11

When $\overset{(1)}{0}$ is transmitted and received $\overset{(1)}{0}$ then probability is $\overline{P} = 1 - P$

When $\underset{(1)}{0}$ is transmitted and $\underset{(0)}{\{1\}}$ is received then probability is P

Occurrence of 0 at i/p is indicated by w and 1 as $\overline{w}$

$$P(x_1) = w$$

$$P(x_2) = \overline{w} = 1 - w$$

P -> probability of error

Channel matrix $P(Y/X)$

$$P(Y/X) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & P(y_1/x_1) & P(y_2/x_1) \\ x_2 & P(y_1/x_2) & P(y_2/x_2) \end{array}$$

$$P(Y/X) = \begin{bmatrix} \overline{P} & P \\ P & \overline{P} \end{bmatrix}$$

13/9

Module-4

Error control coding

All codes will have equal length.

Eg:- Recived triplets

Decoded message.

Disadvantage of having linear variable lengths is any error that occurs during transmission cannot be corrected.

Redundant bits are also called as check bits.

Redundant bits are added either in the beginning or at the end of the code in the linear block codes.

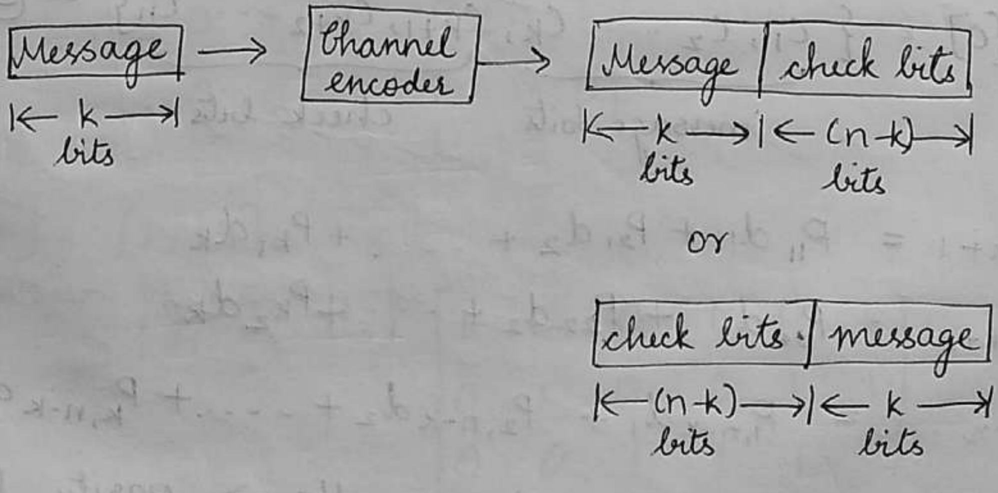
Types of codes

2 types

i) **Block codes** :- Redundant bits are added either in the beginning or at the end - Linear block code

ii) ~~Conventional~~ codes :- Interleaving check bits
Convolution

Linear block codes



Matrix description of linear block code

13/9

Let message block of k bits be represented by a row-vector or k-tuple called "message-vector" given by $[D] = [d_1, d_2 \dots d_k]$ — ①

Thus there are 2^k distinct message vectors.

The channel encoder systematically adds (n-1) no. of check bits to form (n, k) linear block code.

- - -

(n, k) Total no. of messages = n

k no. of message bits

(n-k) no. of bits are check bits

- - -

Then 2^k code vectors can be represented by

$C = \{ C_1, C_2 \dots C_n \}$ — ②

Only k tuples out of n-tuples in eq ② are valid code vectors and the remaining $(2^n - 2^k)$ invalid code vectors. These code vectors are invalid code vectors. These code vectors form error vectors and the invalid code vectors form error vectors and the ratio $\frac{k}{n}$ is defined as the rate efficiency of (n, k) linear block.

$$[C] = \{ \underbrace{C_1, C_2 \dots C_k}_{\text{message bits}}, \underbrace{C_{k+1}, C_{k+2} \dots C_n}_{\text{check bits}} \} \text{ — ③}$$

$$\left.\begin{aligned} C_{k+1} &= P_{11} d_1 + P_{21} d_2 + \dots + P_{k1} d_k \\ C_{k+2} &= P_{12} d_1 + P_{22} d_2 + \dots + P_{k2} d_k \\ C_n &= P_{1,n-k} d_1 + P_{2,n-k} d_2 + \dots + P_{k,n-k} d_k \end{aligned}\right\} \text{ — ④}$$

$P_{11}, P_{12} \dots$ are 0's or 1's ⇒ parity bits

'+' sign in eq ④ is modulo - 2 addition

0+0	0
0+1	1
1+0	1
1+1	0

13/9

$$\underset{\substack{\downarrow \\ \text{code vector}}}{[c_1\, c_2 \cdots c_n]} = \underset{\substack{\downarrow \\ \text{message matrix} \\ \text{(row matrix)} \\ \text{k no of columns}}}{[d_1\, d_2 \ldots d_k]} \underbrace{\begin{bmatrix} \overbrace{1 & 0 & 0 & \ldots & 0}^{k} & \overbrace{P_{11} & P_{12} & \ldots & P_{1,n-k}}^{n-k} \\ 0 & 1 & 0 & \ldots & 0 & P_{21} & P_{22} & \ldots & P_{2,n-k} \\ 0 & 0 & 1 & \ldots & 0 & P_{31} & P_{32} & \ldots & P_{3,n-k} \\ \vdots & & & & & \vdots & & & \\ 0 & 0 & 0 & \ldots & 1 & P_{k1} & P_{k2} & \ldots & P_{k,n-k} \end{bmatrix}}_{}$$

k no. of rows ← Generator matrix
n no. of columns ↓
identity matrix of order k and parity matrix of order (k , n-k)

$[C] = [D][G]$

$[G] = [I_k \,|\, P]_{k \times n}$

Problems

① For a systematic [6,3] linear block code, the parity matrix is given by $[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ Find all possible code vectors.

Sol^n:- $\overset{n\ \ k}{[6, 3]}$

$[C] = [D][G]$ $\qquad [D] = [d_1, d_2, d_3] \qquad D = 3 = k$

$$[G] = \begin{bmatrix} \overbrace{1 & 0 & 0}^{k = 3\times 3} & \overbrace{1 & 0 & 1}^{[P]} \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$[C] = [D][G]$

$$= [d_1\ d_2\ d_3] \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= [\underset{c_1}{d_1}, \underset{c_2}{d_2}, \underset{c_3}{d_3} \underset{c_4}{(d_1 \oplus d_3)}\ \underset{c_5}{(d_2 \oplus d_3)}\ \underset{c_6}{(d_1 \oplus d_2)}]$$

13/9/19

$2^3 = \underset{\substack{\downarrow \\ \text{no. of codes}}}{8}$

code	Message vector $d_1\ d_2\ d_3$	code vector $c_1\ c_2\ c_3\ c_4\ c_5\ c_6$
C_a	0 0 0	0 0 0 0 0 0
C_b	0 0 1	0 0 1 1 1 0
C_c	0 1 0	0 1 0 0 1 1
C_d	0 1 1	0 1 1 1 0 1
C_e	1 0 0	1 0 0 1 0 1
C_f	1 0 1	1 0 1 0 1 1
C_g	1 1 0	1 1 0 1 1 0
C_h	1 1 1	1 1 1 0 0 0

<u>OR</u>

$$[G] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad [D] = [d_1\ \ d_2\ \ d_3]$$

$$[C] = [D][G]$$

$$= [d_1\ \ d_2\ \ d_3] \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[C] = \Big[\underset{c_1}{(d_1 \oplus d_3)} \ \ \underset{c_2}{(d_2 \oplus d_3)} \ \ \underset{c_3}{(d_1 \oplus d_2)} \ \ \underset{c_4}{d_1} \ \ \underset{c_5}{d_2} \ \ \underset{c_6}{d_3} \Big]$$

13/9

code	Message vector d_1 d_2 d_3	Code vector c_1	c_2	c_3	c_4	c_5	c_6
C_a	0 0 0	0	0	0	0	0	0
C_b	0 0 1	1	1	0	0	0	1
C_c	0 1 0	0	1	1	0	1	0
C_d	0 1 1	1	0	1	0	1	1
C_e	1 0 0	1	0	1	1	0	0
C_f	1 0 1	0	1	1	1	0	1
C_g	1 1 0	1	1	0	1	1	0
C_h	1 1 1	0	0	0	1	1	1

We can except $2^6 = 64$ code vectors but we got 8 code vectors and remaining 56 code-vectors are invalid and have error.

20/9 **Circuit** for encoder

$n = 6$

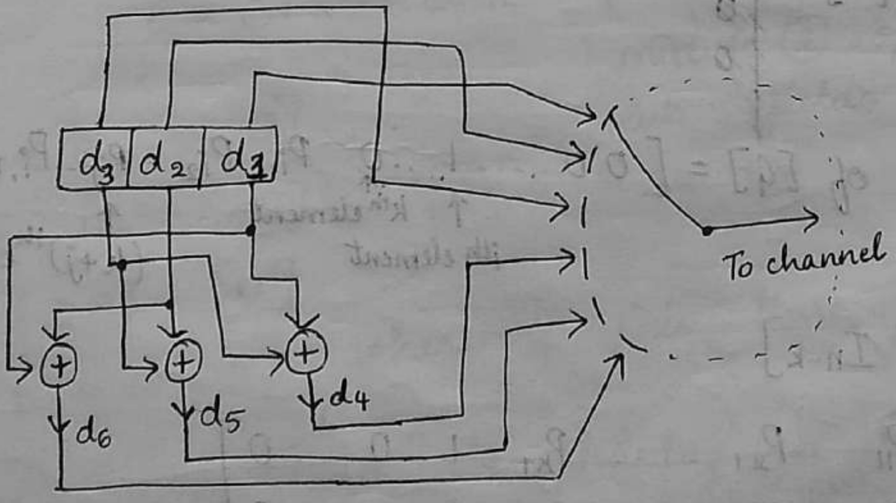
$k = 3$

K bit shift register

$(n-k) \Rightarrow$ modulo-2 adders

n segment commutator

$[d_1, d_2, d_3, (d_1 \oplus d_3), (d_2 \oplus d_3), (d_1 \oplus d_2)]$



20/9

Parity check matrix [H]

$$[H] = [P^T \mid I_{n-k}]$$

$\mid \rightarrow$ separation b/w $\underset{\substack{\downarrow \\ \text{parity matrix}}}{P^T}$ and identity matrix of order n-k

$$[H] = \begin{bmatrix} P_{11} & P_{21} & \cdots & P_{k1} & 1 & 0 & \cdots & 0 \\ P_{12} & P_{22} & \cdots & P_{k2} & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ \underbrace{P_{1,n-k} \quad P_{2,n-k} \quad \cdots \quad P_{k,n-k}}_{k\text{-columns}} & & & & \underbrace{0 \quad 0 \quad \cdots \quad 1}_{(n-k)\text{ columns}} \end{bmatrix}$$

[H] matrix is of order $(n-k) \times n$

Parity check matrix is used for error detection and correction.

~~Proof~~ * If C is a valid code vector then prove that $CH^T = 0$ where H^T is transpose of the parity check matrix H.

Proof:- $C = DG$

$$[C] = [D][G]$$

$$= [D][I_k \mid P]$$

I row of $[G] = [1 \; \overset{(k-1)\text{no. of 0's}}{0 \dots\dots 0} \; P_{11} \; P_{12} \dots P_{1,n-k}]$

i^{th} row of $[G] = [0 \; 0 \dots \underset{\substack{\uparrow \\ i\text{th element}}}{1} \dots \underset{\substack{\uparrow \\ k^{th}\text{element}}}{0} \; P_{i1} \; P_{i2} - \underset{\substack{\uparrow \\ (k+j)^{th}\text{element}}}{P_{ij}} - P_{i,n-k}]$

$$[H] = [P^T \mid I_{n-k}]$$

$$= \begin{bmatrix} P_{11} & P_{21} & \cdots & P_{k1} & 1 & 0 & \cdots & 0 \\ P_{12} & P_{22} & \cdots & P_{k2} & 0 & 1 & \cdots & 0 \\ \vdots & & & & & & & \\ P_{1,n-k} & P_{2,n-k} & \cdots & P_{k,n-k} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

2019

j^{th} row of H matrix $= [\underset{\uparrow\ 1^{st}}{P_{1j}}\ \ P_{2j} \dots \underset{\uparrow\ i^{th}}{P_{ij}} \dots \underset{\uparrow\ k^{th}}{P_{k,j}}\ \ 0\ \ 0 \dots \underset{\uparrow\ (k+j)^{th}\text{ element}}{1} \dots 0]$

$$g_i h_j^T = [0\ 0 \dots \underset{\uparrow\ i^{th}}{1} \dots 0\ \ P_{i1}\ \ P_{i2} \dots \underset{\uparrow\ (k+j)^{th}}{P_{ij}} \dots P_{i,n-k}] \begin{bmatrix} P_{1j} \\ P_{2j} \\ \vdots \\ P_{ij} \\ \vdots \\ P_{kj} \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \rightarrow i^{th} \\ \\ \\ \\ \\ \\ \rightarrow (k+j)^{th} \\ \\ \\ \end{array}$$

$g_i h_j^T$ has ↓ $\underline{n}$ col ↳ $\underline{n}$ rows

$$g_i h_j^T = 0 + 0 + 1 \cdot P_{ij} + 0 + \dots + P_{ij} + \dots 0$$

$$= P_{ij} + P_{ij}$$

$$= P_{ij}(1+1) \Rightarrow \text{modulo 2 addition}$$

$$g_i h_j^T = 0$$

$$\boxed{CH^T = 0}$$

Syndrome and error correction

Syndrome is a bit or a set of bits which indicates exactly where the error has occurred.

$C = [C_1\ C_2\ ----\ C_n] \Rightarrow$ Transmitted vector

$R = [r_1\ r_2\ ----\ r_n] \Rightarrow$ Received vector

$R \neq C \Rightarrow$ Error has occurred

Error $E = C + R = C - R$ [since we are performing module ② operation, add & sub yields error (same result)]

$E = [0\ 0\ ----\ 0]$

$E = [0\ 1\ ----\ 0]$

$E = [0\ 0\ 1 \dots\ 0]$

$E = [1\ ----\ 0]$ wherever 1 is there, that bit ~~is~~ indicates an error.

Error vector will be having n bits

If all bits of error vector is 0 then $R = C$

To identify which bit is causing error and also indicates if error is occurred or not. 2019

$$\therefore \quad S = RH^T$$

Syndrome vector = (Received vector)(Transpose of parity check matrix)

$$S = (S_1, S_2 \ldots S_{n-k})$$

Total no. of bits in syndrome vector is (n-k) bits

$$E = C+R = C-R$$

$$\therefore R = C+E = C-E$$

$$S = RH^T$$

$$S = (C+E)H^T = CH^T + EH^T \quad [\because CH^T = 0]$$

$$\boxed{S = EH^T}$$

Problems

① Referring to (6,3) linear block code of previous problem, the received vector R = [1 1 0 0 1 0]. Detect and correct the single error that has occurred due to noise.

Sol:- $S = RH^T$

$$I_{n-k} = I_{6-3} = I_3$$

$$H = [P^T \,\vdots\, I_{n-k}]$$

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad P^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$S = RH^T$ $\qquad$ $S \rightarrow (n-k)$ no. of bits $\qquad$ 2019

$$= [1\,1\,0\,0\,1\,0]_{1\times 6} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{6\times 3} = [\overset{\overset{1+1=0}{\downarrow}}{1}\ \overset{\overset{1+1=0}{\downarrow}}{0}\ 0]$$

} modulo-2 add^n

$S = 100$

In H^T matrix 100 is the 4th row. ∴ There is a error in received vector in 4th bit from left.

$R = [1\,1\,0\,0\,1\,0] \rightarrow$ this is one among 56 invalid code vector

$E = 0\,0\,0\,1\,0\,0$

$C = R + E = \underline{1\,1\,0\,1\,1\,0} \rightarrow C_g$

If the 4th bit from left is 0 convert to 1; if 1 convert to 0.

Syndrome calculation circuit

Let the received vector $R = (r_1\ r_2 - - - r_n)$. The syndrome $[S] = [S_1\ S_2 \cdots S_{n-k}] = RH^T$

$$[S_1\ S_2 \cdots S_{n-k}] = [r_1\ r_2 - - - r_n] \begin{bmatrix} P_{11} & P_{12} \cdots & P_{1,n-k} \\ P_{21} & P_{22} \cdots & P_{2,n-k} \\ \vdots & & \\ P_{K1} & P_{K2} \cdots & P_{K,n-k} \\ 1 & 0 \cdots & 0 \\ 0 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & 1 \end{bmatrix}$$

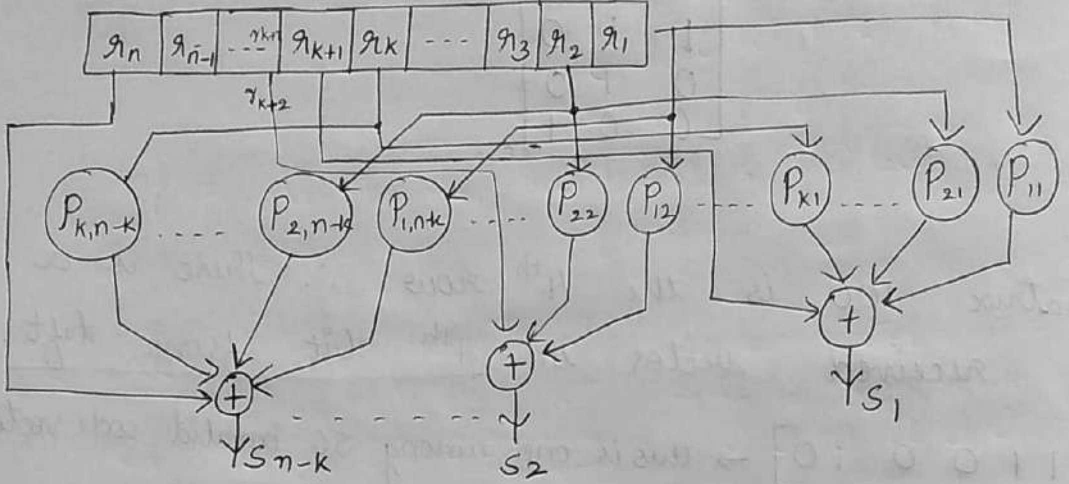
$$S_1 = r_1 P_{11} + r_2 P_{21} + \cdots + r_K P_{K1} + r_{K+1}$$

$$S_2 = r_1 P_{12} + r_2 P_{22} + \cdots + r_K P_{K2} + r_{K+2}$$

$\vdots$

$S_{n-k} = r_1 P_{1,n-k} + r_2 P_{2,n-k} + --- + r_k P_{k,n-k} + r_n$ 23/9

24/9



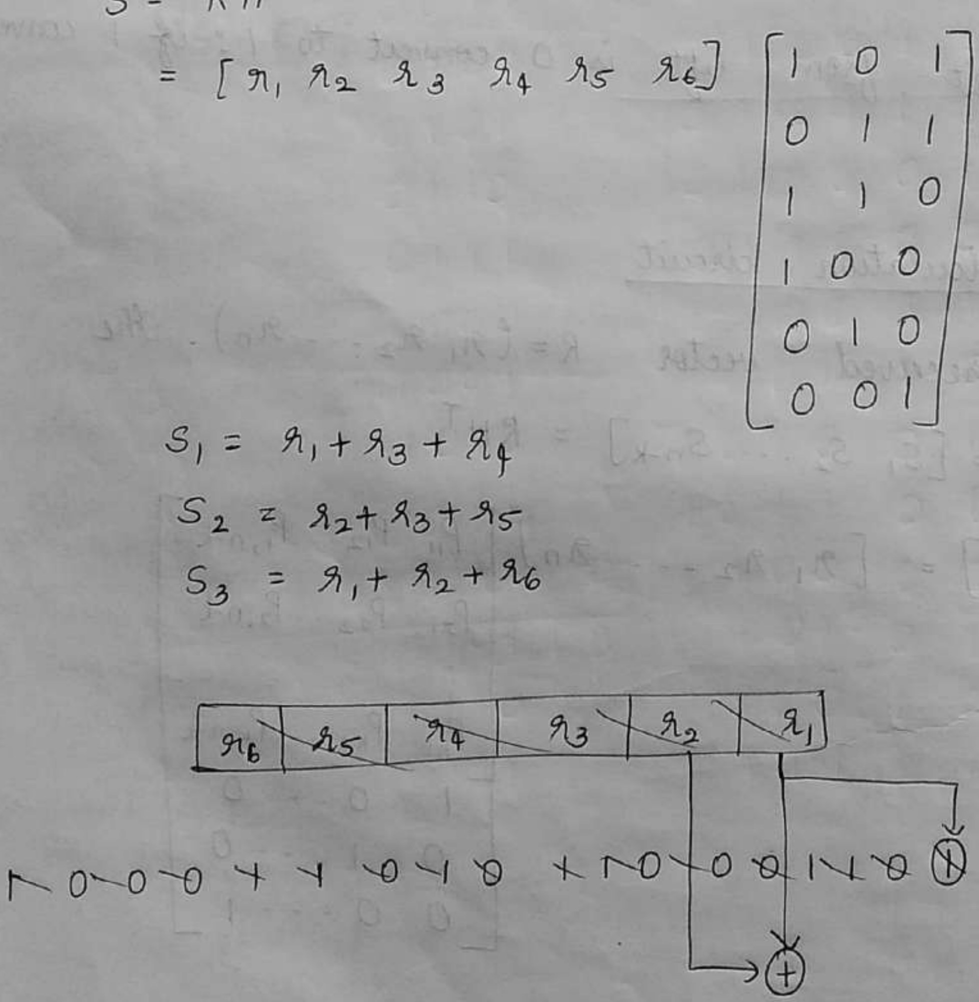
$$S = RH^T$$

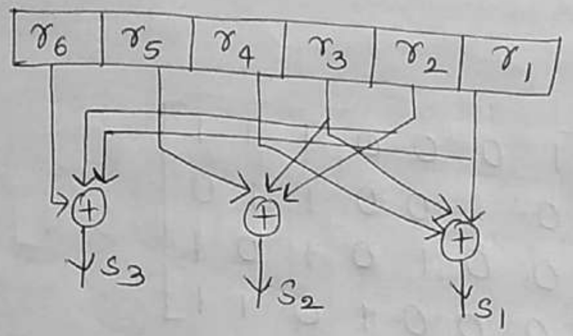
$$= [r_1 \; r_2 \; r_3 \; r_4 \; r_5 \; r_6] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S_1 = r_1 + r_3 + r_4$

$S_2 = r_2 + r_3 + r_5$

$S_3 = r_1 + r_2 + r_6$





24/9

Steps

→ Given (n, k) with Parity matrix

→ Generator matrix $[G] = (k \times n)$ $\quad G = [I \mid P] \quad G = [P \mid I]$

→ $C = [D][G]$ $\quad H = [P^T \mid I_{n-k}]. \; H = [I_{n-k} \mid P^T]$

→ Parity check matrix $H = [P^T \mid I] = (n-k) \times n$

→ syndrome $S = RH^T = (n-k)$ bits

Problem

① For a systematic (7, 4) linear block code, the parity matrix P is given by

$$[P] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(i) Find all possible valid code vectors

(ii) Draw the corresponding encoding circuit

(iii) A single error has occurred in each of these received vectors detect and correct those errors.

(a) $R_A = [0111110]$

(b) $R_B = [1011100]$

(c) $R_C = [1010000]$

(iv) Draw the syndrome calculation circuit.

Sol^n:- [7, 4] $\quad n = 7 \quad k = 4$

$[C] = [D][G]$ $\quad [D] = [d_1 \; d_2 \; d_3 \; d_4]$

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

24/9

$$[C] = [D][G]$$

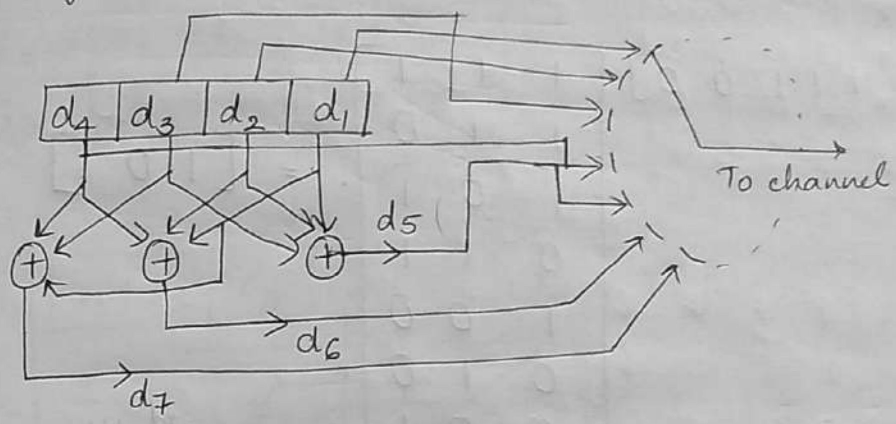
$$= [d_1 \ d_2 \ d_3 \ d_4] \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$= [\underset{c_1}{d_1}, \underset{c_2}{d_2}, \underset{c_3}{d_3}, \underset{c_4}{d_4}, \underset{c_5}{d_1 \oplus d_2 \oplus d_3}, \underset{c_6}{d_1 \oplus d_2 \oplus d_4}, \underset{c_7}{d_1 \oplus d_3 \oplus d_4}]$$

$2^4 = 16$

Code	Message vector $d_1 \ d_2 \ d_3 \ d_4$	code vector $c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6 \ c_7$
C_a	0000	0 0 0 0 0 0 0
C_b	0001	0 0 0 1 0 1 1
C_c	0010	0 0 1 0 1 0 1
C_d	0011	0 0 1 1 1 1 0
C_e	0100	0 1 0 0 1 1 0
C_f	0101	0 1 0 1 1 0 1
C_g	0110	0 1 1 0 0 1 1
C_h	0111	0 1 1 1 0 0 0
C_i	1000	1 0 0 0 1 1 1
C_j	1001	1 0 0 1 1 0 0
C_k	1010	1 0 1 0 0 1 0
C_l	1011	1 0 1 1 0 0 1
C_m	1100	1 1 0 0 0 0 1
C_n	1101	1 1 0 1 0 1 0
C_o	1110	1 1 1 0 1 0 0
C_p	1111	1 1 1 1 1 1 1

Encoding ckt



$S = RH^T$

$H = [P^T \mid I_{n-k}]$ $\qquad I_{7-4} = I_3$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad P^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$S = RH^T$

$$= [0\ 1\ 1\ 1\ 1\ 1\ 0] \begin{bmatrix} 1 & 1 & 1 \\ \boxed{1 \quad 1 \quad 0} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5,7} = [1\ 1\ 0]$$

↓ 2^{nd} bit

a) $R = [0\ 1\ 1\ 1\ 1\ 1\ 0]$

$E = 0\ 1\ 0\ 0\ 0\ 0\ 0$

$C = 0\ 0\ 1\ 1\ 1\ 1\ 0$ ⇒ corrected vector

24/9

b) $S = RH^T$

$$= [1\ 0\ 1\ 1\ 1\ 0\ 0]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ \boxed{1 \quad 0 \quad 1} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{matrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{matrix} = [1\ 0\ 1]$$

3^{rd} bit

$R = 1\ 0\ 1\ 1\ 1\ 0\ 0$
$E = 0\ 0\ 1\ 0\ 0\ 0\ 0$
$C = 1\ 0\ 0\ 1\ 1\ 0\ 0$ ⇒ corrected vector

c) $S = RH^T$

$$= [1\ 0\ 1\ 0\ 0\ 0\ 0]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ \boxed{0 \quad 1 \quad 0} \\ 0 & 0 & 1 \end{bmatrix}\begin{matrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} = [0\ 1\ 0]$$

6^{th} bit

$R = 1\ 0\ 1\ 0\ 0\ 0\ 0$
$E = 0\ 0\ 0\ 0\ 0\ 1\ 0$
$C = 1\ 0\ 1\ 0\ 0\ 1\ 0$ ⇒ corrected vector

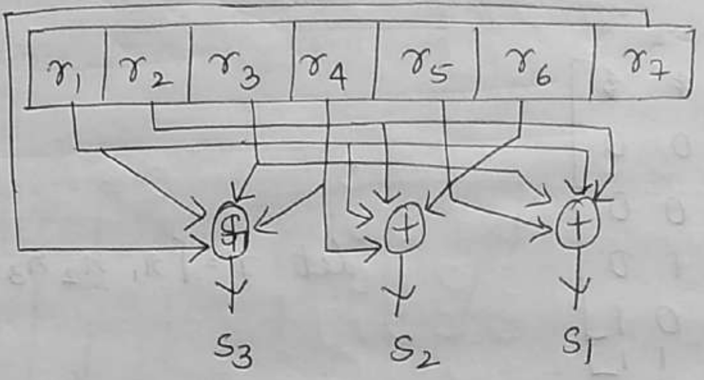
$$S = [r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ r_7]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

24/9

$S_1 = r_1 + r_2 + r_3 + r_5$

$S_2 = r_1 + r_2 + r_4 + r_5$

$S_3 = r_1 + r_3 + r_4 + r_7$



26/9

② Repetition code represents simplest type of linear block codes. The generator matrix of (5,1) repetition code is given by $[G] = [1\ 1\ 1\ 1 \mid 1]$.

i) Write its parity check matrix.

ii) Evaluate the syndrome for all 5 possible single error patterns and also for all 10 possible double error patterns.

Sol^n:- $n = 5$ $k = 1$

$[D] = [d_1\ d_2\ d_3\ d_4\ d_5][1\ 1\ 1\ 1\ 1]$

$= [\underset{c}{d_1}, d_1, d_1, d_1, d_1]$

$2^1 = 2$

Code	Msg vector d_1	Code vector $c_1\ c_2\ c_3\ c_4\ c_5$
C_a	0	0 0 0 0 0
C_b	1	1 1 1 1 1

$[P] = [1\ 1\ 1\ 1]$

$S = RH^T$

$P^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$H = [P^T \mid I_{n-k}]$

$I_{5-1} = I_4$

26/9

$$H = \begin{bmatrix} \cancel{1} & 1 & 0 & 0 & 0 & 1 \\ \cancel{1} & 0 & 1 & 0 & 0 & 1 \\ \cancel{1} & 0 & 0 & 1 & 0 & 1 \\ \cancel{1} & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$G = [P \mid I]$

So $H = [I_{n-k} \mid P^T]$

$$H^T = \begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Let $r = [r_1\ r_2\ r_3\ r_4\ r_5]$

For all possible 5 single error pattern

$S_1 = [1\ 1\ 1\ 1]$

1st bit $R = r_1\ r_2\ r_3\ r_4\ r_5$

$E_1 = 1\ 0\ 0\ 0\ 0$

$C = 0 + 0\ 0\ 0$

$S_2 = [1\ 0\ 0\ 0]$

2nd bit $R = r_1\ r_2\ r_3\ r_4\ r_5$

$E_2 = 0\ 1\ 0\ 0\ 0$

$C = 1\ 1 + 0\ 0$

$S_3 = [0\ 1\ 0\ 0]$

$R = r_1\ r_2\ r_3\ r_4\ r_5$

$E_3 = 0\ 0\ 1\ 0\ 0$

$E_4 = 0\ 0\ 0\ 1\ 0$

$E_5 = 0\ 0\ 0\ 0\ 1$

Error patterns for 5 single

$E_1 = [1\ 0\ 0\ 0\ 0]$

$E_2 = [0\ 1\ 0\ 0\ 0]$

$E_3 = [0\ 0\ 1\ 0\ 0]$

$E_4 = [0\ 0\ 0\ 1\ 0]$

$E_5 = [0\ 0\ 0\ 0\ 1]$

~~Error pattern for~~ 10 double errors

For $S_A = [1\ 0\ 0\ 0]$

$[1\ 0\ 0\ 0\ 0]$

For $[0\ 1\ 0\ 0\ 0]$ $S_B = [0\ 1\ 0\ 0]$

For $[0\ 0\ 1\ 0\ 0]$ $S_C = [0\ 0\ 1\ 0]$

For $[0\ 0\ 0\ 1\ 0]$ $S_D = [0\ 0\ 0\ 1]$

For $[0\ 0\ 0\ 0\ 1]$ $S_E = [1\ 1\ 1\ 1]$

Error pattern for possible 10 double errors.

$E_1 = [1\ 1\ 0\ 0\ 0]$

$E_2 = [0\ 1\ 1\ 0\ 0]$

$E_3 = [0\ 0\ 1\ 1\ 0]$

$E_4 = [0\ 0\ 0\ 1\ 1]$

$E_5 = [1\ 0\ 0\ 0\ 1]$

$E_6 = [1\ 0\ 0\ 1\ 0]$

$E_7 = [1\ 0\ 1\ 0\ 0]$

$E_8 = [0\ 1\ 0\ 0\ 1]$

$E_9 = [0\ 1\ 0\ 1\ 0]$

$E_{10} = [0\ 0\ 1\ 0\ 1]$

$$S_F = E_1 H^T$$

$$= [1\ 1\ 0\ 0\ 0]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\1&1&1&1\end{bmatrix}\begin{matrix}1\\1\\0\\0\\0\end{matrix} = [1\ 1\ 0\ 0]$$

$$S_G = E_2 H^T = [0\ 1\ 1\ 0\ 0]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\1&1&1&1\end{bmatrix}\begin{matrix}0\\1\\1\\0\\0\end{matrix} = [0\ 1\ 1\ 0]$$

$$S_H = E_3 H^T = [0\ 0\ 1\ 1\ 0]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\1&1&1&1\end{bmatrix}\begin{matrix}0\\0\\1\\1\\0\end{matrix} = [0\ 0\ 1\ 1]$$

$$S_I = E_4 H^T = [0\ 0\ 0\ 1\ 1]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\1&1&1&1\end{bmatrix}\begin{matrix}0\\0\\0\\1\\1\end{matrix} = [1\ 1\ 1\ 0]$$

$$S_J = E_5 H^T = [0\ 1\ 1\ 1]$$

$$S_K = E_6 H^T = [1\ 0\ 0\ 1]$$

$$S_L = E_7 H^T = [1\ 0\ 1\ 0]$$

$$S_M = E_8 H^T = [1\ 0\ 1\ 1]$$

$$S_N = E_9 H^T = [0\ 1\ 0\ 1]$$

$$S_O = E_{10} H^T = [1\ 1\ 0\ 1]$$

Hamming weight Hw

26/9

Weight of each code and number of non-zero components in a code.

Eg: $[C_1] = [1\,1\,0\,0\,1\,1\,0]$

Hamming distance

$[C_1] = [\underline{1}\,1\,\underline{0}\,0\,\underline{1}\,1\,\underline{0}]$

$[C_2] = [\underset{\uparrow}{0}\,1\,\underset{\uparrow}{1}\,0\,\underset{\uparrow}{0}\,1\,\underset{\uparrow}{1}]$ ↑ bit is different in c_1 & c_2

There are 4 diff bits.

Hamming distance of C_1 & C_2 is 4.

Minimum distance d_{min}

Smalling hamming distance b/w any 2 code vectors.

(6, 3) linear block code

	code vector
C_a	0 0 0 0 0 0
C_b	0 0 1 1 1 0
C_c	0 1 0 0 1 1
C_d	0 1 1 1 0 1
C_e	1 0 0 1 0 1
C_f	1 0 1 0 1 1
C_g	1 1 0 1 1 0
C_h	1 1 1 0 0 0

Hamming distance

$C_a C_b = 3$ $C_a C_h = 3$

$C_b C_c = 4$ C_a

$C_c C_d = 3$

$C_d C_e = 3$

$C_e C_f = 3$

$C_f C_g = 4$

$C_g C_h = 3$

Minimum distance = 3

26/9

Table lookup decoding (syndrome decoding) using the standard array

Steps for preparing standard array

1. 2^k valid code vectors are placed in a row with all 0 code vectors as the first element.
2. From the remaining $2^n - 2^k$ n_{tuples} and n_{tuple} E_2 is choosen and is placed below all zero code vector. The second row can be formed by placing $(E_2 + C_i)$ under C_i.
3. An unused n_{tuple} E_3 is taken and the third row is completed as given in step 2.
4. The process is continued till all the n_{tuples} are used. The resulting array for an (n, k) linear block code is shown in table below.

$$\begin{array}{lllll}
C_1 = \text{all } 0\text{'s} & C_2 & C_3 & \cdots\cdots & C_{2^k} \\
E_2 & E_2 + C_2 & E_2 + C_3 & \cdots\cdots & E_2 + C_{2^k} \\
E_3 & E_3 + C_2 & E_3 + C_3 & \cdots\cdots & E_3 + C_{2^k} \\
\vdots & & & & \\
E_{2^{n-k}} & E_{2^{n-k}} + C_2 & E_{2^{n-k}} + C_3 & \cdots\cdots & E_{2^{n-k}} + C_{2^k}
\end{array}$$

Problems

① Construct the standard array for (6,3) linear block code given in previous problem.

Syndrome

000000 001110 010011 011101 100101 1·

	1st	2	3	4	5	6	7	8
Syndrome	co-set leader							
000	000000	001110	010011	011101	100101	101011	110110	111000
101	100000	101110	110011	111101	000101	001011	010110	011000
011 only	010000	011110	000011	001101	110101	111011	100110	101000
110 are	001000	000110	011011	010101	101101	100011	111110	110000
100 there	000100	001010	010111	011001	100001	101111	110010	111100
010	000010	001100	010001	011111	100111	101001	110100	111010
001	000001	001111	010010	011100	100100	101010	110111	111001
111	110000	111110	100011	101101	010101	011011	000110	001000

↳ to make n=8 take any one double error

2^{n-k} rows $= 2^{6-3} = 2^3 = 8$ rows

Property of standard array

1. All elements in std array are distinct in nature.
2. First n_{tuple} set of each row is called as co-set leader
3. All 2^k elements in each row have same syndrome

Received vector R = 100100

We found R = 100100 in 7th row 5th column the error pattern is 000001. So add this to R

$$\begin{array}{l} R = 100100 \\ E = 000001 \\ \hline C = 100101 \end{array}$$

C → corrected vector

The first row vector of the column in which received vector is found is the corrected vector.

For Eg if R = 001111 2nd column C = 001110

R = 011011 6th column C = 101011

R = 000011 3rd column C = 010011

R = 011000 8th column C = 111000

27/9

② For the systematic (7, 4) linear block code of problem 2, construct the standard array for the code and express non zero components of co-set leader in terms of syndrome bits S_1, S_2, S_3.

Solⁿ:

$$S_1 = r_1 + r_2 + r_3 + r_5$$
$$S_2 = r_1 + r_2 + r_4 + r_6$$
$$S_3 = r_1 + r_3 + r_4 + r_7$$

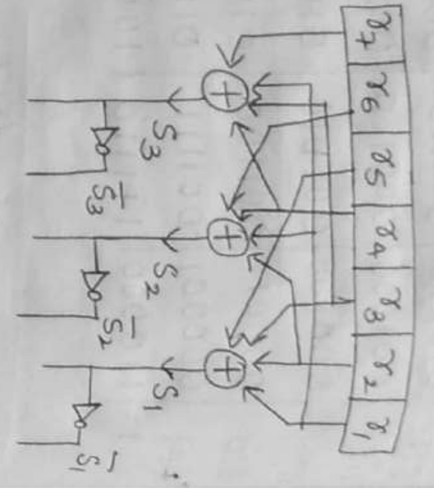
Syndrome	co-set leader									
000	0000000	0001011	0010101	0011110	0100110	0101101	0110011	0111000	1000111	~~1001100~~ ~~1010010~~ 1111111
111	e_1 1000000	1001011	1010101	1011110	1100110	1101101	1110011	1111000	0000111	0111111
110	e_2 0100000	0101011	0110101	0111110	0000110	0001101	0010011	0011000	1100111	1011111
101	0010000	0011011	0000101	0001110	0110110	0111101	0100011	0101000	1010111	1101111
011	0001000	0000011	0011101	- -	- -	- -	- -	- -	- -	1110111
100	0000100	0001111	0010001	- -	- -	- -	- -	- -	- -	1111011
010	0000010	0001001	0010111	- -	- -	- -	- -	- -	- -	1111101
001	0000001	0001010	0010100	- -	- -	- -	- -	- -	- -	1111110
$S_1S_2S_3$										

8 rows

$$2^{n-k} \text{ rows} = 2^{7-4} \text{ rows} = 2^3 = 8 \text{ rows}$$

$$e_1 = S_1 S_2 S_3$$
$$e_2 = S_1 S_2 \overline{S_3}$$
$$e_3 = S_1 \overline{S_2} S_3$$
$$e_4 = \overline{S_1} S_2 S_3$$
$$e_5 = S_1 \overline{S_2}\, \overline{S_3}$$
$$e_6 = \overline{S_1} S_2 \overline{S_3}$$
$$e_7 = \overline{S_1}\, \overline{S_2} S_3$$

Decoding circuit



$\Rightarrow$

General decoding circuit for (n,k) linear block code

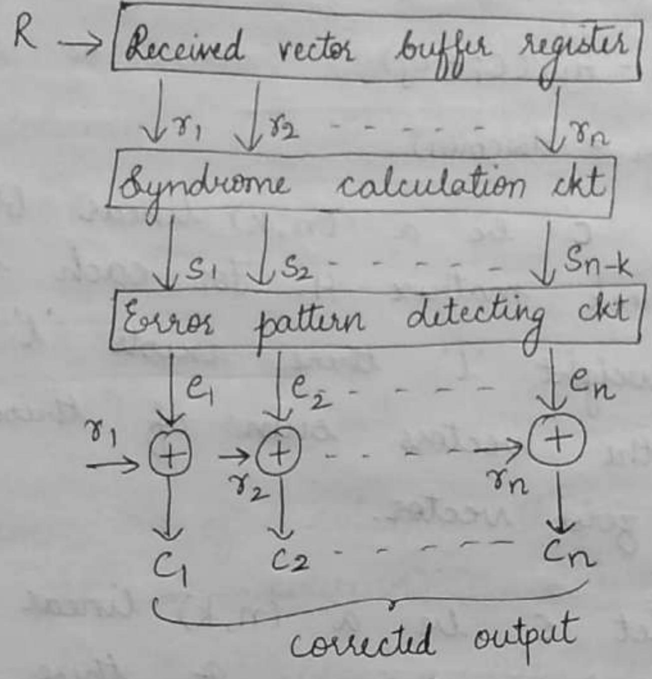
27/9

Decoding circuit : Error detection and correction

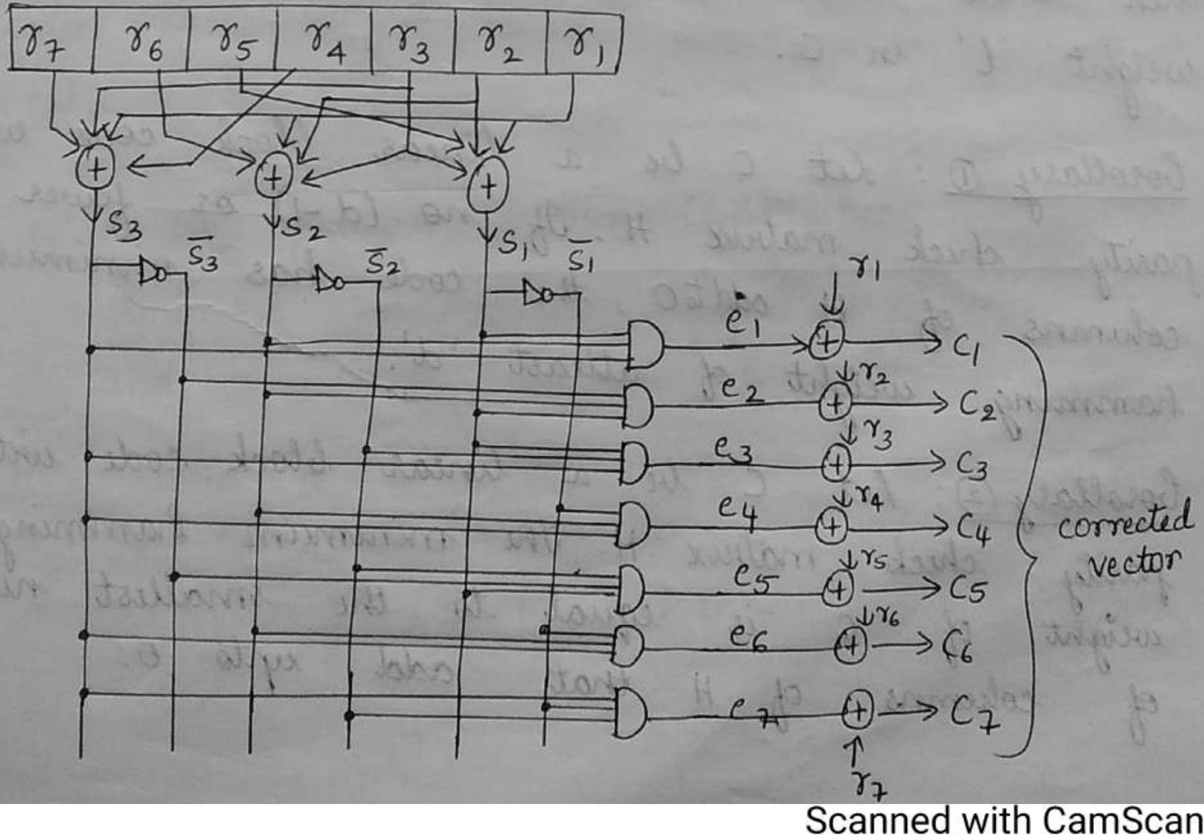
$e_1 = f(S_1\ S_2\ \cdots\ S_{n-k})$

$e_2 = f(S_1\ S_2\ \cdots\ S_{n-k})$

$\dot{e}_n = f(S_1, S_2\ \cdots\ S_{n-k})$



Decoding circuit for (7,4) linear block code.



30/9

Error detecting and error correcting capabilities of linear block code.

Theorem 1 : The minimum distance of a linear block code is equal to the minimum hamming weight of non-zero code vector.

d_{min} = Minimum hamming weight of a non-zero code vector

$$d(C_i, G_i) = H_w (C_i + C_j)$$

$$d_{min} = H_w cmin)$$

Theorem 2(a) : Let C be a (n, k) linear block code with parity check matrix H. For each code vector of hamming weight 'l' there exists 'l' columns of H such that the vectors sum of these 'l' columns is equal to zero vector.

Theorem 2(b) : Let C be a (n, k) linear block code with parity check matrix H. If there exists 'l' columns of H whose vector sum is zero vector, then there exist a code vector of hamming weight 'l' in C.

Corollary ① : Let C be a linear block code with parity check matrix H. If no (d-1) or fewer columns of H add to 0, the code has minimum hamming weight of atleast 'd'.

Corollary ② : Let C be a linear block code with parity check matrix H. The minimum hamming weight of C is equal to the smallest number of columns of H that add upto 0.

30/9

Proof :

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$H = [P^T \,|\, I_{n-k}]$

Since $I = I_3$ and

no. of columns = 7

$n-k=3 \Rightarrow 7-k=3$

$K=4$

So it is (7,4) linear block code.

First add any 2 columns to get 0 column. If we don't get 0 column by adding 2 columns, then add 3 columns to check if we can get 0 column.

For eg:- 1^{st} & 2^{nd} column $= \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} + 7^{th}$ column $\begin{matrix} 0 \\ 0 \\ 1 \end{matrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$

$1+2+7 = 0$

$|||^{ly}$ $2+3+4 = \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} + \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} = \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} + \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$

∴ Minimum no. of columns that are required to add to get 0 column = Minimum hamming weight

Theorem 3 : A linear block code with minimum distance dmin can detect upto (dmin -1) errors and can correct upto [(dmin -1)]/2 errors where $\frac{(dmin-1)}{2}$ indicates the largest integer not greater than $\frac{(dmin-1)}{2}$.

For eg :- For previous problem (6,3) linear block code dmin = 3 it can detect only 2 errors (dmin -1) and it can correct only 1 error $\left[\left(\frac{dmin-1}{2}\right) = \frac{3-1}{2} = 1\right]$

if dmin = 4

It can detect 3 errors and correct upto $\frac{4-1}{2} = \frac{3}{2} = \underset{\sim}{1.5}$ i.e., 1 error

Single error correcting Hamming codes

30/9

$H = [P^T ; I_{n-k}]$ — ①

$H^T = \begin{bmatrix} P \\ -- \\ I_{n-k} \end{bmatrix} \Rightarrow (n-k)$ no. of columns — ②

From the matrix of eq ② we observe that H^T has 'n' no. of rows and $(n-k)$ no. of columns. The condition for all the rows of H^T to be distinct is that $\boxed{2^{n-k} - 1 \geq n}$

We have taken (-1) because there is no 0 row or 0 matrix.

$$2^{n-k} - 1 \geq n$$

$$2^{n-k} \geq n+1$$

$$n-k \geq \log_2(n+1)$$

$$\boxed{k \leq n - \log_2(n+1)}$$

code length : $n \leq 2^{n-k} - 1$

No. of message bits : $k \leq n - \log_2(n+1)$

No. of parity check ~~mabr~~ bits : $n-k$

Error correcting capability : $t = \frac{(d_{min}-1)}{2}$

Problems :

① Design (n, k) hamming code with a min. distance of 3 and message length of 4 bits.

Sol :- Given $k = 4$ $d_{min} = 3$

$$n \leq 2^{n-k} - 1$$

Use trial & error method to find n

30/9

Always $n > k$

Assume $n = 5$ $\quad 5 \leq 2^{5-4} - 1 \Rightarrow 5 \nleq 1$

$n = 6$ $\quad 6 \leq 2^{2} - 1 \Rightarrow 6 \nleq 3$

$n = 7$ $\quad 7 \leq 2^{3} - 1 \Rightarrow 7 \leq 7$

$\therefore \boxed{n = 7}$

(7, 4) linear block code

$$H^T = \begin{bmatrix} P \\ \hline I_{n-k} \end{bmatrix} = \begin{bmatrix} & P & \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$H^T \rightarrow$ 'n' $\rightarrow$ rows

$(n-k) \rightarrow$ columns

$P = (4, 3)$

3 columns

0 0 0
0 0 1
0 1 0
0 1 1 ✓
1 0 0
1 0 1 ✓
1 1 0 ✓
1 1 1 ✓

All columns and rows must be distinct

Among the remaining 4 codes, any code can be written in any order

$$H^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

1/10

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$d_{min} = 3$

$C = [D][G]$

$K = 4$

$[G] = [I_K \,\vdots\, P]$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$[D] = d_1 \, d_2 \, d_3 \, d_4$

1/10

$$C = [D][G]$$

$$= [d_1\ d_2\ d_3\ d_4] \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$C = [\underset{c_1}{d_1},\ \underset{c_2}{d_2},\ \underset{c_3}{d_3},\ \underset{c_4}{d_4},\ \underset{c_5}{d_2 \oplus d_3 \oplus d_4},\ \underset{c_6}{d_1 \oplus d_3 \oplus d_4},\ \underset{c_7}{d_1 \oplus d_2 \oplus d_4}]$$

Code

Message vector				Code vector						
d_1	d_2	d_3	d_4	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	1	1	1
0	0	1	0	0	0	1	0	1	1	0
0	0	1	1	0	0	1	1	0	0	1
0	1	0	0	0	1	0	0	1	0	1
0	1	0	1	0	1	0	1	0	1	0
0	1	1	0	0	1	1	0	0	1	1
0	1	1	1	0	1	1	1	1	0	0
1	0	0	0	1	0	0	0	0	1	1
1	0	0	1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0	1	0	1
1	0	1	1	1	0	1	1	0	01	0
1	1	0	0	1	1	0	0	1	1	0
1	1	0	1	1	1	0	1	0	0	1
1	1	1	0	1	1	1	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1

$$t = \frac{d_{min} - 1}{2} = \frac{3-1}{2} = 1$$

It can detect 2 errors and can correct only 1 error. So it is called single error correcting code (SEC)

Hamming Bound

For (n,k) linear block code there are 2^{n-k} syndrome patterns including all zero syndrome. Each syndrome corresponds to a specific error pattern. If 'i' is the no. of error locations in 'n' dimensional error pattern 'e' in general nC_i multiple error patterns will be there.

∴ The total no. of all possible error patterns $= \sum_{i=0}^{t} nC_i$ where t is the max no. of error locations in 'e'

If an (n,k) linear block code is to be capable of correcting upto 'i' errors, then the total no. of syndromes shall not be less than the total no. of all possible error patterns i.e., $\boxed{2^{n-k} \geq \sum_{i=0}^{t} nC_i}$. This equation is called as Hamming Bound.

The no. of syndromes ≥ no. of errors.

For previous problem

$$t = \frac{d_{min}-1}{2} = 1$$

$$2^{n-k} \geq \sum_{i=0}^{t} nC_i$$

$$2^3 \geq \sum_{i=0}^{1} 7C_i$$

$$8 \geq 7C_0 + 7C_1$$

$$8 \geq 1+7$$

$$8 \geq 8 \qquad 8=8 \quad \Rightarrow \text{Perfect code}$$

(n,k) linear block code which satisfies this inequality with equal sign is called perfect code.

A binary code for which the hamming bound turns out to be equality is called perfect code.

Problems

① The parity check bits of (8,4) block code are generated by $c_5 = d_1 + d_2 + d_4$, $c_6 = d_1 + d_2 + d_3$, $c_7 = d_1 + d_3 + d_4$, $c_8 = d_2 + d_3 + d_4$ where d_1, d_2, d_3 and d_4 are the message bits.

i) Find the generator matrix and parity check matrix for this code.

ii) Find the minimum weight of this code

iii) Show that it is capable of correcting all single error patterns and capable of detecting all double errors by preparing syndrome table for them.

Sol :- (8,4) $n = 8$ $k = 4$

P ~~=~~ $H = [P^T \mid I_{n-k}]$ $I_{\cancel{8}-4} = I_4$

G ~~H~~ $= [I_{K} \mid P]$

$$\cancel{H}\, G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad P^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

1/10

~~$d_{min} = 8$~~

$H = [P^T \mid I_{n-k}]$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H^T = \begin{bmatrix} 1110 \\ 1101 \\ 0111 \\ 1011 \\ 1000 \\ 0100 \\ 0010 \\ 0001 \end{bmatrix}$$

$d_{min} = 4$

$t = \frac{d_{min} - 1}{2} = \frac{3}{2} = 1$ → Corrects single error patterns

$C = [D][G]$

$$C = [d_1\ d_2\ d_3\ d_4] \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$C = [d_1\ d_2\ d_3\ d_4\ \ d_1 \oplus d_2 \oplus d_4\ \ d_1 \oplus d_2 \oplus d_3\ \ d_1 \oplus d_3 \oplus d_4\ \ d_2 \oplus d_3 \oplus d_4]$$

Wt of Code	Message vector	Code vector							
		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
~~0~~ 0	0 0 0 0	0	0	0	0	0	0	0	0
4	0 0 0 1	0	0	0	1	1	0	1	1
4	0 0 1 0	0	0	1	0	0	1	1	1
4	0 0 1 1	0	0	1	1	1	1	0	0
4	0 1 0 0	0	1	0	0	1	1	0	1
4	0 1 0 1	0	1	0	1	0	1	1	0
4	0 1 1 0	0	1	1	0	1	0	1	0
4	0 1 1 1	0	1	1	1	0	0	0	1
4	1 0 0 0	1	0	0	0	1	1	1	0
4	1 0 0 1	1	0	0	1	0	1	0	1
4	1 0 1 0	1	0	1	0	1	0	0	1
4	1 0 1 1	1	0	1	1	0	0	1	0
4	1 1 0 0	1	1	0	0	0	0	1	1
4									

Wt of code		
4	1101	11011000
4	1110	11100100
8	1111	11111111

Minimum weight = 4

Syndrome table   syndrome bit = $n-k = 8-4 = 4$

Single error pattern	Syndrome
10000000	1110
01000000	1101
00100000	0111
00010000	1011
00001000	1000
00000100	0100
00000010	0010
00000001	0001

Syndrome is calculated as $S = RH^T$

Assume any one bit with error

consider $R = 10000000$

$S = RH^T$

$$S = [S1 \ S2 \ S3 \ S4] = [10000000] \begin{bmatrix} 1110 \\ 1101 \\ 0111 \\ 1011 \\ 1000 \\ 0100 \\ 0010 \\ 0001 \end{bmatrix}$$

$S = [1110] \rightarrow 1^{st}$ row

1^{st} bit error

$$\begin{aligned} R &= 10000000 \\ E &= 10000000 \\ \hline C &= 00000000 \end{aligned}$$

Double error pattern	Syndrome
11000000	0011
10100000	1001
⋮	⋮
01100000	1010
⋮	⋮
00011000	0011
⋮	⋮
000010010	1001
⋮	⋮
000000~~0~~11	0011

$$H^T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ & \vdots & & \\ 1 & 0 & 1 & 0 \\ & \vdots & & \\ 0 & 0 & 1 & 1 \\ & \vdots & & \\ 1 & 0 & 0 & 1 \\ & \vdots & & \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 1 & \cdots & 1 & & 0 & 1 & 0 \\ 0 & 0 & & 0 & \cdots & 0 & \cdots\, 0 & \cdots\, 0 \\ 1 & 0 & & 1 & & 1 & 0 & 1 \\ 1 & 1 & & 0 & & 1 & 1 & 1 \end{bmatrix}$$

$\therefore d_{min} = 2$

~~No.~~ of Detecting capability $= d_{min} - 1 = 2 - 1 = 1$

Problems

① Design a (4, 2) linear block code

i) Find generator matrix for the code vector set

ii) Find parity check matrix

iii) Choose the code vectors to be in systematic form with the goal of maximising dmin.

iv) Enter the 16 4 tuples in a standard array

v) What are the error detecting and correcting capabilities of the code?

vi) Make a syndrome table for the correctable error patterns.-

vii) Draw the encoding circuit.

viii) Draw the syndrome calculating circuit.

② The parity check bits of a (7, 4) hamming code are generated by $C_5 = d_1 + d_3 + d_4$, $C_6 = d_1 + d_2 + d_3$, $C_7 = d_2 + d_3 + d_4$ where d_1, d_2, d_3 and d_4 are the message bits.

i) Find the generator matrix G and parity check matrix H for this code.

ii) Prove that $G X H^T = 0$

iii) The (n, k) linear block code so obtained as a dual code, this dual code is (n, n-k) code having the generator matrix H and parity check matrix G. Determine the 8 code vectors of the dual code for the (7, 4) hamming code described above.

iv) Find the minimum distance of dual code determined in iii)

Binary Cyclic codes

Cyclic codes are subclass of linear block code

Advantages

i) Encoding and syndrome calculating circuits are simpler compared to LBC and can be implemented using shift registers, feedback ckts using basic gates.

ii) They have mathematical (algebraic) structure and is used to ~~correct~~ implement error correcting circuits.

Eg:- $C_1 = 11000110$

Then the other code vectors of the same code are the shifted (cyclic) version of C_1 i.e

C_2 may be $= 00011011$

or

C_2 may be $= 10001101$

$C_3 = 00011011$

Modulo 2 Algebra

Addition:

$x + x = x(1+1) = x(0) = 0$

$x - x = x(1-1) = 0$

Addition and subtraction is same in modulo 2 algebra.

3/10

Multiplication :

$x . x = x^2$; $x^2 . x = x^3$; $x^3 . x = x^4$; $x^2 . x^2 = x^4$

Problems

① Find the product of polynomials $f_1(x) = x+1$ and $f_2(x) = x^3+x+1$ using modulo-2 algebra.

Sol :- product $= f_1(x) . f_2(x)$

$$f_1(x) . f_2(x) = (x+1)(x^3+x+1)$$

$$= x^4 + x^2 + x + x^3 + x + 1 \qquad [\because x+x=0]$$

$$= x^4 + x^3 + x^2 + 0 + 1$$

$$= x^4 + x^3 + x^2 + 1$$

② $f_1(x) = 1 + x + x^3$ $\qquad f_2(x) = 1 + x + x^2 + x^4$

Sol :- $f_1(x) . f_2(x) = (1+x+x^3)(1+x+x^2+x^4)$

$$= 1 + x + x^2 + x^4 + x + x^2 + x^3 + x^5 + x^3 + x^4 + x^5 + x^7$$

$$= x^7 + x^5(1+1) + x^4(1+1) + x^3(1+1) + x^2(1+1) + x(1+1) + 1$$

$$= x^7 + 0 + 0 + 0 + 0 + 0 + 1$$

$$= x^7 + 1$$

③ Divide $f_2(x) = x^6 + x^5 + x^2$ by $f_1(x) = x^3 + x + 1$ by modulo 2 algebra.

Sol :-

$$x^3+x+1) \; x^6 + x^5 + x^2 \; (x^3 + x^2 + x$$

$$\underline{\underset{(-)}{x^6} \underset{(-)}{+ x^4} \underset{(-)}{+ x^3}}$$

$$+ x^5 + x^4 + x^3 + x^2$$

$$\underline{\underset{(-)}{x^5} - \cancel{x} \underset{(+)}{+ x^3} \underset{(-)}{+ x^2}}$$

$$+ x^4$$

$$\underset{(+)}{+x^4} \underset{(+)}{+x^2} \underset{(+)}{+x}$$

$$\overline{\quad x^2 + x \quad} \Rightarrow \text{Remainder polynomial } R(s)$$

$\Uparrow$ quotient polynomial $Q(x)$

+ & - are same in modulo 2.

3/10

④ If $f(x) = x^4+x+1$ then show that $[f(x)]^2 = f(x^2)$ in modulo 2 algebra.

Sol :- $[f(x)]^2 = (x^4+x+1)(x^4+x+1)$

$= x^8+x^5+x^4+x^5+x^2+x+x^4+x+1$

$[f(x)]^2 = x^8+x^2+1$

$f(x^2) = (x^2)^4+x^2+1$

$[f(\dot{x})] = x^8+x^2+1$

Hence showed $f(x^2) = [f(x)]^2$

This equation is true for all polynomials of any degree.

4/10

$C_1 = 0111001$

$C_2 = 1011100$

$C_3 = 0101110$

$C_4 = 0010111$

$V = (V_0\ V_1\ V_2 \cdots V_{n-1})$ — ①

If V belongs to cyclic codes, then

$$\left.\begin{aligned} V^{(1)} &= (V_{n-1}\ V_0\ V_1 \cdots V_{n-2}) \\ V^{(2)} &= (V_{n-2}\ V_{n-1}\ V_0\ V_1 \cdots V_{n-3}) \\ &\vdots \end{aligned}\right\} -②$$

In general $V^{(i)} = (V_{n-i}\ V_{n-i+1} \cdots V_0\ V_1 \cdots V_{n-i-1})$

Expressing eq ① in polynomial form

$V(x) = V_0 + V_1x + V_2x^2 + \cdots + V_{n-1}x^{n-1}$ — ③

$$\left.\begin{aligned} \text{lly}\quad V^{(1)}(x) &= V_{n-1} + V_0x + V_1x^2 + \cdots + V_{n-2}x^{n-1} \\ V^{(2)}(x) &= V_{n-2} + V_{n-1}x + V_0x^2 + \cdots + V_{n-3}x^{n-1} \\ &\vdots \\ \text{In general}\quad V^{(i)}(x) &= V_{n-i} + V_{n-i+1}x + \cdots + V_{n-i-1}x^{n-1} \end{aligned}\right\} -④$$

$V_0, V_1 \cdots V_{n-1}$ belong to binary field.

* Properties of cyclic codes

4/10

1. For a (n,k) cyclic code, there exists a generator polynomial of degree (n-k) given by

$$g(x) = g_0 + g_1 x + g_2 x^2 + \dots + g_{n-k} x^{n-k} \text{ — ⑤}$$

2. The generator polynomial g(x) of (n,k) cyclic code is a factor of $x^{n}+1$ i.e., $x^{n}+1 = g(x) \cdot h(x)$ — ⑥ where h(x) is a another polynomial of degree k and is called parity check polynomial.

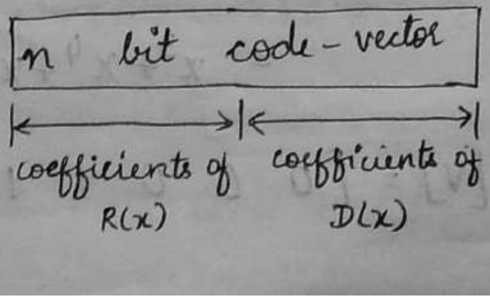
3. If g(x) is a polynomial of degree (n-k) and is a factor of x^n+1, then it generates (n,k) cyclic code.

4. The code vector polynomial V(x) can be found by using $V(x) = \cancel{d}\, D(x) \cdot g(x)$ — ⑦ where D(x) is a message vector polynomial of degree k.

$$D(x) = d_0 + d_1 x + d_2 x^2 + \dots + d_{k-1} x^{k-1} \text{ — ⑧}$$

D(x) is a multiple of g(x). It generates non systematic cyclic code

5. To generate systematic cyclic code the remainder polynomial R(x) is obtained from division of $x^{n-k} D(x)$ by g(x). The coefficients of R(x) are placed in the beginning of code vector followed by the coefficients of message polynomial D(x) to get the code vector.



4/10

Problems

① For the (7,4) single error correcting cyclic code $D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$ and $x^n + 1 = x^7 + 1 = [(1+x+x^3) \times (1+x+x^2+x^4)]$. Using the generator polynomial $g(x) = 1+x+x^3$, find all the 16 code vectors of cyclic code both in systematic and non-systematic form.

Sol^n:-

For Non systematic cyclic code

$$V(x) = D(x)g(x)$$

Let $D = [1011]$

$$D(x) = d_0 + d_2 x^2 + d_3 x^3$$

$$D(x) = 1 + x^2 + x^3$$

$$V(x) = (1+x^2+x^3)(1+x+x^3)$$

$$= 1 + x + x^3 + x^2 + x^3 + x^5 + x^3 + x^4 + x^6$$

$$= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

$$= 1 + 1.x + 1.x^2 + 1.x^3 + 1.x^4 + 1.x^5 + 1.x^6$$

$$[V] = [1\ 1\ 1\ 1\ 1\ 1\ 1]$$

$D = [0\ 0\ 0\ 0]$

$$[V] = [0\ 0\ 0\ 0\ 0\ 0\ 0]$$

$D = [0\ 0\ 0\ 1]$ $\qquad V(x) = x^3(1+x+x^3)$

$D(x) = d_3 x^3 = x^3$ $\qquad = x^3 + x^4 + x^6$

$$[V] = [0\ 0\ 0\ 1\ 1\ 0\ 1]$$

$D = [0010] \qquad D(x) = x^2$

$$V(x) = x^2(1+x+x^3) = x^2+x^3+x^5$$

$$[V] = [0\ 0\ 1\ 1\ 0\ 1\ 0]$$

$D = [0011] \qquad D(x) = x^2+x^3$

$$V(x) = (x^2+x^3)(1+x+x^3) = x^2+x^3+x^5+x^3+x^4+x^6$$

$$= x^2+x^4+x^5+x^6$$

$$[V] = [0\ 0\ 1\ 0\ 1\ 1\ 1]$$

$D = [0100] \qquad D(x) = x$

$$V(x) = x(1+x+x^3) = x+x^2+x^4$$

$$[V] = [0\ 1\ 1\ 0\ 1\ 0\ 0]$$

$D = [0101] \qquad D(x) = x+x^3$

$$V(x) = (x+x^3)(1+x+x^3) = x+x^2+x^4+x^3+x^4+x^6$$

$$= [x+x^2+x^3+x^6]$$

$$[V] = [0\ 1\ 1\ 1\ 0\ 0\ 1]$$

$D = [0110] \qquad D(x) = x+x^2$

$$V(x) = (x+x^2)(1+x+x^3) = x+x^2+x^4+x^2+x^3+x^5$$

$$= x+x^3+x^4+x^5$$

$$[V] = [0\ 1\ 0\ 1\ 1\ 1\ 0]$$

$D = [0111] \qquad D(x) = x+x^2+x^3$

$$V(x) = (x+x^2+x^3)(1+x+x^3) = x+x^2+x^4+x^2+x^3+x^5+x^3+x^4+x^6$$

$$= x+x^5+x^6$$

$$[V] = [0\ 1\ 0\ 0\ 0\ 1\ 1]$$

$D = [1000] \qquad D(x) = 1$

$V(x) = 1(1+x+x^3) = 1+x+x^3$

$[V] = [1\ 1\ 0\ 1\ 0\ 0\ 0]$

$D = [1001] \qquad D(x) = 1+x^3$

$V(x) = (1+x^3)(1+x+x^3) = 1+x+x^3+x^3+x^4+x^6$

$= 1+x+x^4+x^6$

$[V] = [1\ 1\ 0\ 0\ 1\ 0\ 1]$

$D = [1010] \qquad D(x) = 1+x^2$

$V(x) = (1+x^2)(1+x+x^3) = 1+x+x^3+x^2+x^3+x^5$

$= 1+x+x^2+x^5$

$[V] = [1\ 1\ 1\ 0\ 0\ 1\ 0]$

$D = [1011] \qquad D(x) = 1+x^2+x^3$

$V(x) = (1+x^2+x^3)(1+x+x^3) = 1+x+x^3+x^2+x^3+x^5+$
$x^3+x^4+x^6$

$= 1+x+x^2+x^3+x^4+x^5+x^6$

$[V] = [1\ 1\ 1\ 1\ 1\ 1\ 1]$

$D = [1100] \qquad D(x) = 1+x$

$V(x) = (1+x)(1+x+x^3) = 1+x+x^3+x+x^2+x^4$

$= 1+x^2+x^3+x^4$

$[V] = [1\ 0\ 1\ 1\ 1\ 0\ 0]$

$D = [1\ 1\ 0\ 1]$ $\quad D(x) = 1 + x + x^3$

$V(x) = (1+x+x^3)(1+x+x^3) = 1+x+x^3+x+x^2+x^4$
$+ x^3+x^4+x^6$

$= 1+x^2+x^6$

$[V] = [1\ 0\ 1\ 0\ 0\ 0\ 1]$

$D = [1\ 1\ 1\ 0]$ $\quad D(x) = 1+x+x^2$

$V(x) = (1+x+x^2)(1+x+x^3) = 1+x+x^3+x+x^2+x^4+$
$x^2+x^3+x^5$

$= 1+x^4+x^5$

$[V] = [1\ 0\ 0\ 0\ 1\ 1\ 0]$

$D = [1\ 1\ 1\ 1]$ $\quad D(x) = 1+x+x^2+x^3$

$V(x) = (1+x+x^2+x^3)(1+x+x^3) = 1+x+x^3+x+x^2+$
$x^4+x^2+x^3+x^5+x^3+x^4+x^6$

$= 1+x^3+x^5+x^6$

$[V] = [1\ 0\ 0\ 1\ 0\ 1\ 1]$

Message vector	Code vector	Message vector	Code vector
0000	0000000	1000	1101000
0001	0001101	1001	1100101
0010	0011010	1010	1110010
0011	0010111	1011	1111111
0100	0110100	1100	1011100
0101	0111001	1101	1010001
0110	0101110	1110	1000110
0111	0100011	1111	1001011

4/10

Systematic cyclic code

$$\boxed{\frac{x^{n-k}\,D(x)}{g(x)} = g(\cancel{x})\ R(x)}$$

$x^{7-4} = x^3$

Let $D(x) = 1 + x + x^3$ for $D = [1101]$

$x^{n-k}\,D(x) = x^3(1+x+x^3) = x^3+x^4+x^6$

$n-k = \textcircled{3}$

$$\begin{array}{l} x^3+x+1\ \big)\ x^6+x^4+x^3\ \big(\ x^3 \\ \qquad\qquad\ \ \underline{x^6+x^4+x^3} \\ \qquad\qquad\qquad\ 0 \end{array}$$

$R(x)=0$ $\quad R = 000$

$V = [0\ 0\ 0\ 1\ 1\ 0\ 1]$

$D = [1\ 0\ 0\ 1]$ $\quad D(x) = 1+x^3$

$x^3(1+x^3) = x^3+x^6$

$$\begin{array}{l} x^3+x+1\ \big)\ x^6+x^3\ \big(\ x^3+x \\ \qquad\qquad\ \ \underline{x^6+x^4+x^3} \\ \qquad\qquad\quad x^4+ \\ \qquad\qquad\quad \underline{x^4+x^2+x} \\ \qquad\qquad\qquad x^2+x \end{array}$$

$R(x) = x+x^2$

$[R] = [0\ 1\ 1]$

$V = [\underbrace{0\ 1\ 1}_{R}\ \underbrace{1\ 0\ 0\ 1}_{M}]$

$D = [0\ 0\ 0\ 0]$ $\quad R(x)=0$ $\quad V = [0\ 0\ 0\ 0\ 0\ 0\ 0]$

$D = [0,0\ 0\ 1]$ $\quad D(x) = x^3$

$x^3(x^3) = x^6$

$$\begin{array}{l} x^3+x+1\ \big)\ x^6\ \qquad \big(\ x^3+x+1 \\ \qquad\qquad\ \ \underline{x^6+x^4+x^3} \\ \qquad\qquad\quad x^4+x^3 \\ \qquad\qquad\quad x^4+x^2+x \ = \ x^3+x^2+x \end{array}$$

$$\begin{array}{l} x^3+x^2+x \\ \underline{x^3+x+1} \\ \quad x^2+1 \end{array}$$

$R(x) = x^2+1 \qquad R = [1\ 0\ 1]$

$[V] = [1\ 0\ 1\ 0\ 0\ 0\ 1]$

$D = [0\ 0\ 1\ 0] \qquad D = x^2$

$$\frac{x^3(x^2)}{x^3+x+1} = R(x)$$

$$\begin{array}{r l}
x^3+x+1) & \cancel{x^5} \quad (x^2+1 \\
 & \underline{\cancel{x^5}+x^3+x^2} \\
 & \quad \cancel{x^3}+x^2 \\
 & \quad \underline{\cancel{x^3}+x+1} \\
 & \qquad x^2+x+1
\end{array}$$

$R(x) = x^2+x+1$

$R = [1\ 1\ 1]$

$[V] = [1\ 1\ 1\ 0\ 0\ 1\ 0]$

$D = [0\ 0\ 1\ 1] \qquad D = x^2+x^3$

$x^3(x^2+x^3) = x^5+x^6$

$$\begin{array}{r l}
x^3+x+1) & \cancel{x^6}+x^5 \quad (x^3+x^2+x \\
 & \underline{\cancel{x^6}+x^4+x^3} \\
 & \quad \cancel{x^5}+x^4+x^3 \\
 & \quad \underline{\cancel{x^5}+\cancel{x^3}+x^2} \\
 & \qquad \cancel{x^4}+\cancel{x^2} \\
 & \qquad \underline{\cancel{x^4}+\cancel{x^2}+x} \\
 & \qquad\quad x
\end{array}$$

$R(x) = x$

$R = [0\ 1\ 0]$

$[V] = [0\ 1\ 0\ 0\ 0\ 1\ 1]$

$D = [0\ 1\ 0\ 0] \qquad D = x$

$x^3(x) = x^4$

$$\begin{array}{r l}
x^3+x+1) & \cancel{x^4} \quad (x \\
 & \underline{\cancel{x^4}+x^2+x} \\
 & \qquad x^2+x
\end{array}$$

$R(x) = x^2+x$

$[R] = [0\ 1\ 1]$

$\cancel{R} \quad [V] = [0\ 1\ 1\ 0\ 1\ 0\ 0]$

4|10

$D = [0\ 1\ 0\ 1]$ $\qquad D(x) = x + x^3$

$$x^3(x+x^3) = x^4 + x^6$$

$$x^3+x+1)\ \cancel{x^6}+\cancel{x^4}\ (x^3+1$$
$$\underline{x^6+x^4+x^3}$$
$$\cancel{x^3}.$$
$$\underline{\cancel{x^3}+x+1}$$
$$x+1$$

$R(x) = x+1$

$[R] = [1\ 1\ 0]$

$[V] = [1\ 1\ 0\ 0\ 1\ 0\ 1]$

$D = [0\ 1\ 1\ 0]$ $\qquad D(x) = x + x^2$

$$x^3(x+x^2) = x^4 + x^5$$

$$x^3+x+1)\ \cancel{x^5}+x^4\ (x^2+x+1$$
$$\underline{x^5+x^3+x^2}$$
$$\cancel{x^4}+x^3+\cancel{x^2}$$
$$\underline{x^4+x^2+x}$$
$$\cancel{x^3}+x$$
$$\underline{x^3+\cancel{x}+1}$$
$$1$$

$R(x) = 1$

$[R] = [1\ 0\ 0]$

$[V] = [1\ 0\ 0\ 0\ 1\ 1\ 0]$

$D = [0\ 1\ 1\ 1]$ $\qquad D(x) = x + x^2 + x^3$

$$x^3(x+x^2+x^3) = x^4 + x^5 + x^6$$

$$x^3+x+1)\ \cancel{x^6}+x^5+\cancel{x^4}\ (x^3+x^2$$
$$\underline{\cancel{x^6}+\cancel{x^4}+x^3}$$
$$\cancel{x^5}+\cancel{x^3}$$
$$\underline{\cancel{x^5}+\cancel{x^3}+x^2}$$
$$x^2$$

$R(x) = x^2$

$[R] = [0\ 0\ 1]$

$[V] = [0\ 0\ 1\ 0\ 1\ 1\ 1]$

$D = [1\ 0\ 0\ 0]$ $\qquad D(x) = 1$

$x^3(1) = x^3$

$$x^3+x+1)\ \cancel{x^3}\ (1$$
$$\underline{\cancel{x^3}+x+1}$$
$$x+1$$

$R(x) = x+1$

$[R] = [1\ 1\ 0]$

$[V] = [1\ 1\ 0\ 1\ 0\ 0\ 0]$

$D = [1\ 0\ 1\ 0]$ $D(x) = 1 + x^2$ 4|10

$x^3(1+x^2) = x^3 + x^5$

$x^3+x+1)\ x^5 + x^3\ (x^2$

$\underline{x^5 + x^3 + x^2}$

x^2

$R(x) = x^2$

$[R] = [0\ 0\ 1]$

$[V] = [0\ 0\ 1\ 1\ 0\ 1\ 0]$

$D = [1\ 0\ 1\ 1]$ $D(x) = 1 + x^2 + x^3$

$x^3(1+x^2+x^3) = x^3 + x^5 + x^6$

$x^3+x+1)\ x^6 + x^5 + x^3\ (x^3 + x^2 + x + 1$

$\underline{x^6 + x^4 + x^3}$

$x^5 + x^4$

$\underline{x^5 + x^3 + x^2}$

$x^4 + x^3 + x^2$

$\underline{x^4 + x^2 + x}$

$x^3 + x$

$\underline{x^3 + x + 1}$

1

$R(x) = \cancel{x}\ 1$

$[R] = [100]$

$[V] = [100\ 1\ 0\ 1\ 1]$

$D = [1\ 1\ 0\ 0]$ $D(x) = 1 + x$

$x^3(1+x) = x^3 + x^4$

$x^3+x+1)\ x^4 + x^3\ (x + 1$

$\underline{x^4 + x^2 + x}$

$x^3 + x^2 + x$

$\underline{x^3 + x + 1}$

$x^2 + 1$

$R(x) = 1 + x^2$

$[R] = [1\ 0\ 1]$

$[V] = [1\ 0\ 1\ 1\ 1\ 0\ 0]$

$D = [1\ 1\ 0\ 1]$ $D(x) = 1 + x + x^3$

$x^3(1+x+x^3) = x^3 + x^4 + x^6$

$x^3+x+1)\ x^6 + x^4 + x^3\ (x^3$

$\underline{x^6 + x^4 + x^3}$

0

$[R] = 0\ 0\ 0$

$[V] = [0\ 0\ 0\ 1\ 1\ 0\ 1]$

4/10

$D = [1\ 1\ 1\ 0]$ $\qquad D(x) = 1+x+x^2$

$$x^3(1+x+x^2) = x^3+x^4+x^5$$

$$x^3+x+1\ )\ x^5+x^4+x^3\ (\ x^2+x$$

$$x^5+x^3+x^2$$

$$x^4+x^2$$

$$x^4+x^2+x$$

$$x$$

$R(x) = x$

$[R] = [0\ 1\ 0]$

$[V] = [0\ 1\ 0\ 1\ 1\ 1\ 0]$

$D = [1\ 1\ 1\ 1]$ $\qquad D(x) = 1+x+x^2+x^3$

$$x^3(1+x+x^2+x^3) = x^3+x^4+x^5+x^6$$

$$x^3+x+1\ )\ x^6+x^5+x^4+x^3\ (\ x^3+x^2+1$$

$$x^6+x^4+x^3$$

$$x^5$$

$$x^5+x^3+x^2$$

$$x^3+x^2$$

$$x^3+x+1$$

$$x^2+x+1$$

$R(x) = x^2+x+1$

$[R] = [1\ 1\ 1]$

$[V] = [1\ 1\ 1\ 1\ 1\ 1\ 1]$

Message vector	Code vector	Message vector	Code vector
0000	0000000	1000	1101000
0001	1010001	1001	0111001
0010	1110010	1010	0011010
0011	0100011	1011	1001011
0100	0110100	1100	1011100
0101	1100101	1101	0001101
0110	1000110	1110	0101110
0111	0010111	1111	1111111

Parity check matrix H

10/10

$x^n+1 = g(x)h(x)$

For (7,4) cyclic codes $n=7$ $x^7+1 = g(x)h(x)$

h(x) is called as parity check polynomial

$\boxed{h(x) = \frac{x^7+1}{g(x)}}$ $h(x) = \frac{x^{n}+1}{g(x)}$ $\boxed{h(x) = \frac{x^n+1}{g(x)}}$

$g(x) = 1+x+x^3$

$$\begin{array}{l} x^3+x+1)\ x^7+1\ (x^4+x^2+x+1 \\ \quad\quad\quad\ \underline{x^7+x^5+x^4} \\ \quad\quad\quad\quad x^5+x^4+1 \\ \quad\quad\quad\quad \underline{x^5+x^3+x^2} \\ \quad\quad\quad\quad\quad x^4+x^3+x^2+1 \\ \quad\quad\quad\quad\quad \underline{x^4+x^2+x} \\ \quad\quad\quad\quad\quad\quad x^3+x+1 \\ \quad\quad\quad\quad\quad\quad \underline{x^3+x+1} \\ \quad\quad\quad\quad\quad\quad\quad\quad 0 \end{array}$$

h(x) is the quotient polynomial

$h(x) = x^4+x^2+x+1 = 1+x+x^2+x^4$

The reciprocal of h(x) is defined as $\boxed{x^k h(x^{-1})}$ and this polynomial is also a factor of $1+x^n$

For (7,4) cyclic codes $k=4$

Let us consider $x^4h(x^{-1})$

The 2 cyclic shifted versions of $x^4h(x^{-1})$ are $x^5h(x^{-1})$ and $x^6h(x^{-1})$

$h(x) = 1+x+x^2+x^4$

$h(x^{-1}) = 1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^4}$

$x^4h(x^{-1}) = x^4+x^3+x^2+1$

$= 1+0.x+1.x^2+1.x^3+1.x^4+0.x^5+0.x^6$

1011100

10/10

$x^5 h(x^{-1}) \Rightarrow$ similar to cyclically shifted version

$$x^5 h(x^{-1}) = x^5 + x^4 + x^3 + x = x + x^3 + x^4 + x^5$$

0101110

$$x^6 h(x^{-1}) = x^6 + x^5 + x^4 + x^2 = x^2 + x^4 + x^5 + x^6$$

0010111

$$[H] = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

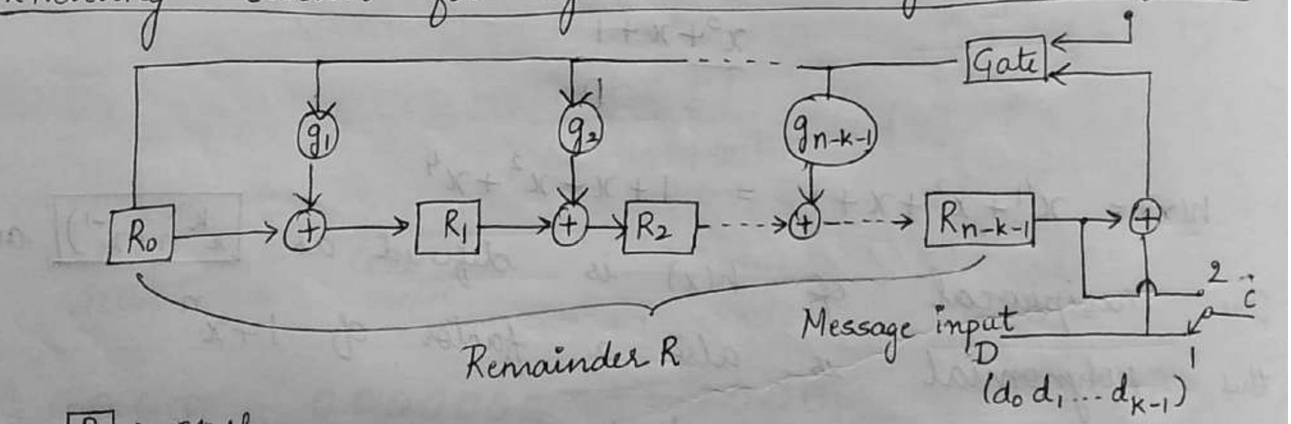
[H] should be in either $I_{n-k} ; P^T$ or $P^T ; I_{n-k}$

In order to get I_{n-k} modify the matrix by matrix calculations

$R_1 \to R_1 + R_3$

$$[H] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} = [I_{n-k} ; P^T]$$

Encoding circuit for cyclic code using shift registers



$\boxed{R} \to$ Flipflops

ⓖ → represents 1 (closed path) or 0 (open path)

$\boxed{\text{Gate}} \to$ AND gate

Connecting flipflops together represents shift register.

It is $(n-k)$ bit shift register

$$\underbrace{R_0 R_1 \ldots R_{n-k-1}}^{R} \ \underbrace{d_0 d_1 \ldots d_{k-1}}^{D}$$

$(n-k)$ bits $\quad k$ bits

⊕ ⇒ Modulo 2 adder

$(n-k)$ number of flipflops are required

10/10

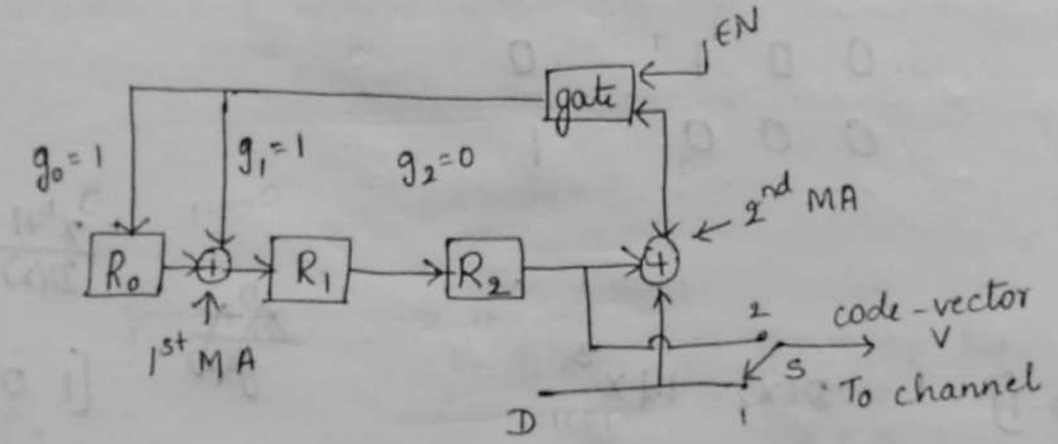
① For a (7,4) cyclic code design an encoder ckt for $g(x) = 1+x+x^3$ and verify its operation using message vectors 1001, 1011, 1111, 1101

Solⁿ:- $g(x) = 1+x+x^3$

$$g(x) = g_0 + g_1(x) + g_2x^2 + \dots + g_{n-k}x^{n-k}$$

$$g(x) = (g_0=1) + 1.x + 1.x^3 = 1 + 1.x + 0.x^2 + 1.x^3$$

$g_0 = 1$ $g_1 = 1$ $g_2 = 0$ $g_3 = 1$



No. of shifts	Input D	Shift reg contents R_0 R_1 R_2	Remainder bits R
Gate is on and switch is in position 1		0 0 0	–
1	1	1 1 0	–
2	0	0 1 1	–
3	0	1 10 1	–
4	1	0 1 1	
5	–	0 0 1	1
6	–	R_a R_1 R_2	
7	~~1~~	0 0 0	1
	~~0~~	0 0 0	0
	~~1~~		

Switch will be moved to position 2

For the 1st shift whever was there in R_0 in previous clk cycle will be added with R_1

11/10

1011

No. of shifts	D	R_0	R_1	R_2	Remainder bit R
		0	0	0	
1	1	1	1	0	–
2	01	01	0	1	–
3	0	1	0	0	–
4	1	1	0	0	–
5	–	0	1	0	0
6	–	0	0	1	0
7	–	0	0	0	1

Verification

11/10

$D = [1\ 0\ 0\ 1]$ $\quad D(x) = 1 + x^3$

$x^3(1+x^3) = x^3 + x^6$

$x^3+x+1\ )\ x^6+x^3\ (\ x^3+x$

$x^6+x^4+x^3$

x^4

x^4+x^2+x

x^2+x

$[0\ 1\ 1]$

$D = [1\ 0\ 1\ 1]$

$D(x) = 1+x^2+x^3$

$x^3(1+x^2+x^3) = x^6+x^5+x^3$

$x^3+x+1\ )\ x^6+x^5+x^3\ (\ x^3+x^2+x+1$

$x^6+x^4+x^3$

x^5+x^4

$x^5+x^3+x^2$

$x^4+x^3+x^2$

x^4+x^2+x

x^3+x

x^3+x+1

1

$[1\ 0\ 0]$

$[1\ 1\ 1\ 1]$

No. of shifts	D	$R_0\ R_1\ R_2$	Remainder bit R
		0 0 0	
1	1	1 1 0	
2	1	1 0 1	
3	1	0 1 0	
4	1	1 1 1	
5	-	0 1 1	1
6	-	0 0 1	1
7	-	0 0 0	1

$D = [1\ 1\ 1\ 1]$

$x^3(1+x+x^2+x^3) = x^3+x^4+x^5+x^6$

$x^3+x+1\ )\ x^6+x^5+x^4+x^3\ (\ x^3+x^2+1$

$x^6+x^4+x^3$

x^5

$x^5+x^3+x^2$

x^3+x^2

x^3+x+1

x^2+x+1

$[1\ 1\ 1]$

11/10

$D = [1\,1\,0\,1]$

No. of shifts	D	R_0	R_1	R_2	Remainder bits
		0	0	0	
1	1	1	1	0	-
2	0	0	1	1	-
3	1	0	0	1	-
4	1	0	0	0	-
5	-	0	0	0	0
6	-	0	0	0	0
7	-	0	0	0	0

$[1\,1\,0\,1]$

$$D(x) = 1 + x + x^3$$

$$x^3(1+x+x^3) = x^3 + x^4 + x^6$$

$$x^3+x+1 \overline{) \, x^6 + x^4 + x^3} \; (\, x^3$$

$$\underline{x^6 + x^4 + x^3}$$

$$\underline{0}$$

$[0 \;\; 0 \;\; 0]$

Syndrome calculation - Error detection and ~~calculation~~ correction

$V(x)$ is the transmitted vector.

$Z(x)$ is the received vector.

$$\frac{Z(x)}{g(x)} = Q(x) + \frac{S(x)}{g(x)} \quad — ①$$

$Q(x) \rightarrow$ quotient polynomial

$S(x) \rightarrow$ syndrome polynomial

The degree (order) of syndrome polynomial is $(n-k-1)$

If there is any error in received vector, 11/10
then it can be corrected by

$Z(x) = V(x) + E(x)$ — ②

$V(x) = Z(x) + E(x)$ — ③

$V(x) = g(x) D(x)$ — ④

$Z(x) = D(x) g(x) + E(x)$ — ⑤

$\frac{Z(x)}{g(x)} = D(x) + \frac{E(x)}{g(x)}$ — ⑥

Substituting eq ⑥ in eq ①

$$D(x) + \frac{E(x)}{g(x)} = Q(x) + \frac{S(x)}{g(x)}$$

$$\frac{E(x)}{g(x)} = Q(x) + D(x) + \frac{S(x)}{g(x)}$$

$E(x) = g(x)[\,Q(x) + D(x)] + S(x)$ — ⑦

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INFORMATION THEORY AND CODING

Problems

① The expurgated (n, K-1) Hamming code is obtained from the original (n, k) Hamming code by discarding some of the code-vectors. Let g(x) denote the generator polynomial of the original Hamming code. The most common expurgated Hamming code is the one generated by $g_1(x) = (1+x)g(x)$ where $(1+x)$ is a factor of $1+x^n$.

Consider the (7,4) Hamming code generated by

$$g(x) = 1+x^2+x^3$$

a) Construct the 8 code vectors in the expurgated (7,3) Hamming code, assuming a systematic format. Hence, show that the minimum distance of the code is 4.

b) Determine the generator matrix G_1 and the parity-check matrix H_1 of the expurgated Hamming code.

c) Devise the encoder for the expurgated Hamming code and list the shift register contents in a tabular fashion for the message 011. Verify the code-vector so obtained using $[V] = [D]^T[G_1]$.

d) Device the syndrome calculator for the expurgated Hamming code. Hence, determine the syndrome for the received vector 0111110. Also correct the error, if any, in that received vector.

Solⁿ:- In an expurgated Hamming code, the number of message bits is reduced by 1 bit but the no. of check bits is increased by 1 bit so that the total bit length remains at n. Such a Hamming code is called as (n, k-1) expurgated Hamming code.

$$g_1(x) = (1+x)g(x)$$
$$= (1+x)(1+x^2+x^3)$$
$$= 1+x^2+x^3+x+x^3+x^4$$

$$g_1(x) = 1+x+x^2+x^4$$

For $g_1(x) = 1+x+x^2+x^4$ code → 1110100

For $x g_1(x) = x+x^2+x^3+x^5$ code → 0111010

For $x^2 g_1(x) = x^2+x^3+x^4+x^6$ code → 0011101

$$[G_1] = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$r_3 \rightarrow r_1 + r_3$$

$$[G_1] = \left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$[G_1] = [P \; I_3]$$

$$H_1 = [I_{n-(k-1)} \,\vdots\, P^T]$$

$$= [I_4 \,;\, P^T]$$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$[V] = [D][G]$$

$$= [d_0 \; d_1 \; d_2] \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= [d_0 \oplus d_2, \; d_0 \oplus d_1 \oplus d_2, \; d_0 \oplus d_1, \; d_1 \oplus d_2, \; d_0, \; d_2, \; d_2]$$

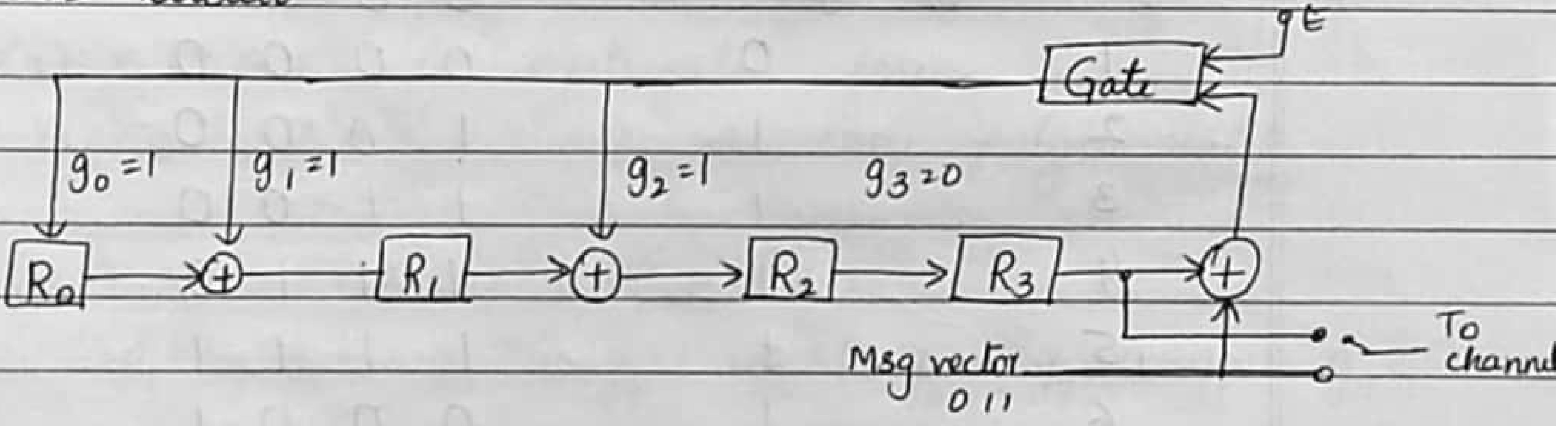
Msg vector $d_0\ d_1\ d_2$	Code vector $c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7$	Hamming wt H_w
0 0 0	0 0 0 0 0 0 0	0
0 0 1	1 1 0 1 0 0 1	4
0 1 0	0 1 1 1 0 1 0	4
0 1 1	1 0 1 0 0 1 1	4
1 0 0	1 1 1 0 1 0 0	4
1 0 1	0 0 1 1 1 0 1	4
1 1 0	1 0 0 1 1 1 0	4
1 1 1	0 1 0 0 1 1 1	4

$d_{min} = 4$

(c) $g_1(x) = 1 + x + x^2 + x^4$

$g_0 = 1 \quad g_1 = 1 \quad g_2 = 1 \quad g_3 = 0 \quad g_4 = 1$

Encoder circuit



For a message of 011, the shift register contents are

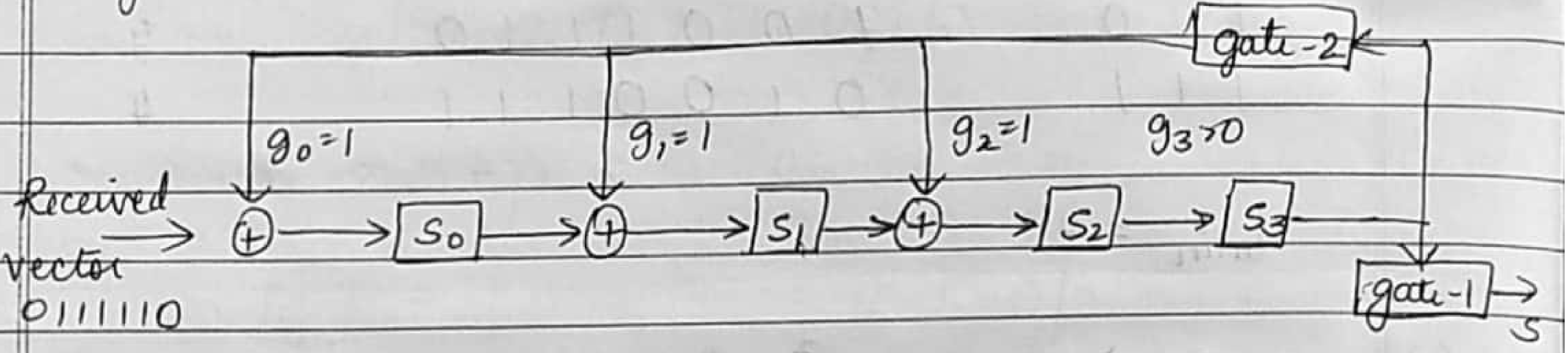
No. of shifts	Input D	Shift register content R_0	 R_1	 R_2	 R_3	Remainder bits R
		0	0	0	0	-
1	1	1	1	1	0	-
2	1	1	0	0	1	-
3	0	1	0	1	0	-
4	x	0	1	0	1	0
5	x	0	0	1	0	1
6	x	0	0	0	1	0
7	x	0	0	0	0	1

∴ The code vector going into the channel is 1010011 which verifies that it is a valid code vector.

d) $g_1(x) = 1 + x + x^2 + x^4$

$g_0 = 1 \quad g_1 = 1 \quad g_2 = 1 \quad g_3 = 0$

Syndrome calculator circuit



No of shifts	Input	Shift register content				Comments
	Z(x)	S_0	S_1	S_2	S_3	
gate -1 off	gate 2 -on	0	0	0	0	Shift register contents are cleared
1	0	0	0	0	0	
2	1	1	0	0	0	
3	1	1	1	0	0	
4	1	1	1	1	0	
5	1	1	1	1	1	
6	1	0	0	0	1	
7	0	1	1	1	0	← Indicates error
8	0	0	1	1	1	
9	0	1	1	0	1	
10	0	1	0	0	0	← End of shifting operation

The syndrome of received vector is 1110. To correct the error 0's are fed into the register till the contents of the shift register read 1000, the first row of H^T From 10^{th} shift we get 1000. The error is located and corrected as given below.

The received vector $z \rightarrow 0111110$

↑ ↑ ↑

10th 9th 8th shift.

Since we got 1000 in the 10th shift, the 3rd bit from right is in error

$$E = 0000100$$

$$V = Z + E = \begin{array}{r} 0111110 \\ 0000100 \\ \hline 0111010 \end{array} \Rightarrow \text{Corrected vector}$$

② A (15,5) linear cyclic code has a generator polynomial

$$g(x) = 1 + x + x^2 + x^4 + x^5 + x^8 + x^{10}$$

a) Draw the block diagram of an encoder and syndrome calculator for this code.

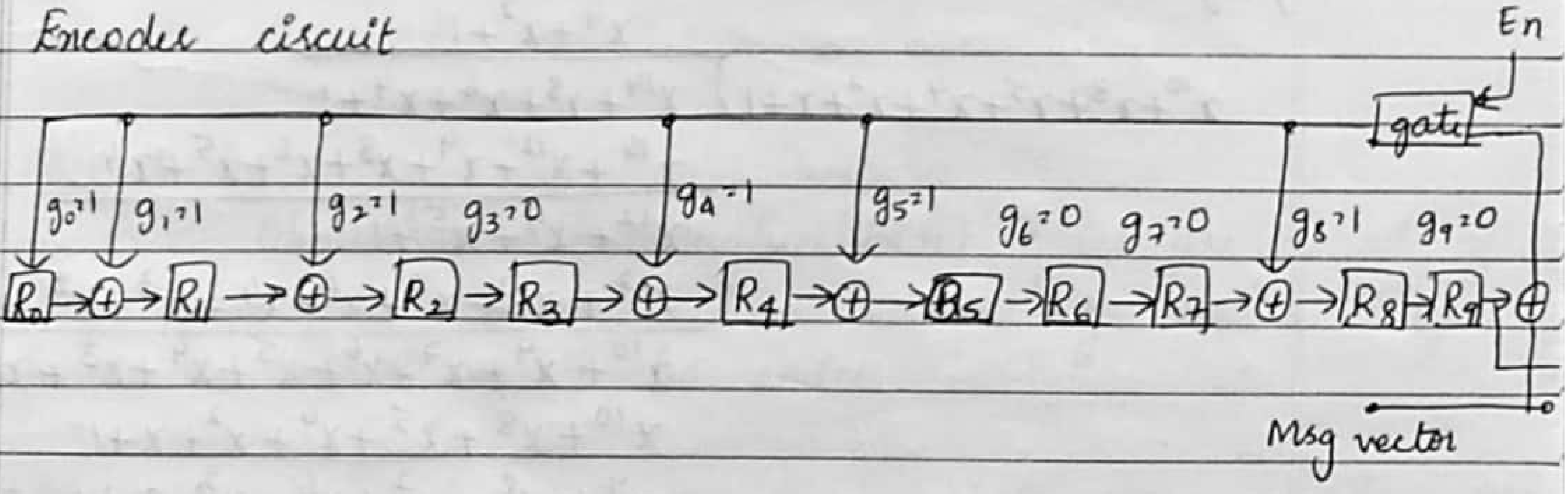
b) Find the code polynomial for the message polynomial $D(x) = 1 + x^2 + x^4$ in systematic form

c) Is $V(x) = 1 + x^4 + x^6 + x^8 + x^{14}$ a code polynomial?

Sol^n a) $g(x) = 1 + x + x^2 + x^4 + x^5 + x^8 + x^{10}$

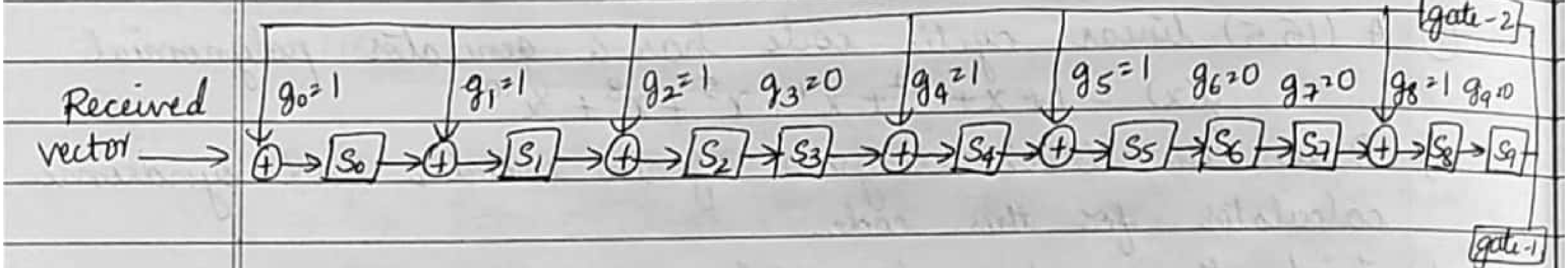
$g_0 = 1 \quad g_1 = 1 \quad g_2 = 1 \quad g_3 = 0 \quad g_4 = 1 \quad g_5 = 1 \quad g_6 = 0 \quad g_7 = 0 \quad g_8 = 1 \quad g_9 = 0$

Encoder circuit



No. of shifts	Input D	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R
		0	0	0	0	0	0	0	0	0	0	
1	1	1	1	1	0	1	1	0	0	1	0	-
2	0	0	1	1	1	0	1	1	0	0	1	-
3	1	0	0	1	1	1	0	1	1	0	0	-
4	0	0	0	0	1	1	1	0	1	1	0	-
5	1	1	1	1	0	0	0	1	0	0	1	-

Syndrome calculator circuit



c) If $V(x)$ is a code-polynomial, then it should be perfectly divisible by the generator polynomial with remainder 0. If the remainder is not 0, we can conclude that the given polynomial is not a code-polynomial.

$$x^4+x^2+1$$

$$x^{10}+x^8+x^5+x^4+x^2+x+1 \,\big)\, x^{14}+x^8+x^6+x^4+1$$

$$\underline{x^{14}+x^{12}+x^9+x^8+x^6+x^5+x^4}$$

$$x^{12}+x^9+x^5+1$$

$$\underline{x^{12}+x^{10}+x^7+x^6+x^4+x^3+x^2}$$

$$x^{10}+x^9+x^7+x^6+x^5+x^4+x^3+x^2$$

$$\underline{x^{10}+x^8+x^5+x^4+x^2+x+1}$$

$$x^9+x^8+x^7+x^6+x^3+x+1 \Rightarrow R \neq 0$$

Hence given polynomial is not a code polynomial.

Module - 5

Golay Codes

Binary Golay codes are one of the very important classes of linear block codes, since they are one among the few sets of non-trivial perfect codes. In 1949, Golay discovered a possibility of the combination (23,12) for the construction for a perfect code. The (23,12) Golay codes are capable of correcting 3 errors. Any code which satisfies the following Hamming Bound is called a perfect code:

$$2^{n-k} = \sum_{i=0}^{t} {}^{n}C_i$$

$n = 23$ $k = 12$ the above equation is satisfied. Thus Golay codes are class of perfect codes.

The following are the parameters for a Golay codes.

Number of code bits $n = 23$
Number of data bits $k = 12$
Number of parity bits $(n-k) = 11$
Number of error correcting capability $t = 3$.
Generator polynomial $g(x) = 1 + x^2 + x^4 + x^5 + x^6 + x^{10} + x^{11}$
or $g(x) = 1 + x + x^5 + x^6 + x^7 + x^9 + x^{11}$
Hence $d_{min} = 7$

BCH Codes

Bose - Chaudhuri - Hocquenghen (BCH) codes are a set of very powerful random error correcting codes. They form a subset of cyclic codes.

For any +ve integer $m \geq 3$, there exists a BCH code with
Block length, $n = 2^m - 1$
Parity bits $(n-k) \leq mt$

Minimum distance $d_{min} \geq 2t+1$

The code obtained by the above said parameters will lead us to a t error correcting BCH codes. As BCH codes form a subclass of cyclic codes, they can be characterised by its generator polynomial. The generator polynomial $g(x)$ for a BCH code can be defined as the minimal monic polynomial over GF(2) which has $\alpha, \alpha^2, \alpha^3 \ldots \alpha^{2t}$ as roots, where α is a primitive element in $GF(2^m)$. that is,

$$g(\alpha^4) = 0 \quad 1 \leq i \leq 2t$$

$g(x)$ has $\alpha, \alpha^2, \alpha^3 \ldots \alpha^{2t}$ and their conjugates as roots. Let $\phi_i(x)$ denote the minimal polynomial α^i. Then

$$g(x) = LCM[\phi_1(x), \phi_2(x), \ldots \phi_{2t}(x)]$$

If 'i' is an even number, it can be expressed as

$$i = i_1 2^l$$

where i_1 is an odd number and l is an integer, $l \geq 1$. Then $\alpha^i = \alpha^{i_1 2^l} = (\alpha^{i_1})^{2^l}$ fall under the conjugacy class of α^{i_1}.

$$\therefore \quad g(x) = LCM[\phi_1(x), \phi_3(x) \ldots \phi_{2t-1}(x)]$$

As the degree of minimal polynomial is m or less the generator polynomial degree can be at most equal to mt. ∴ The parity bits (n-k) can be at most equal to mt.

Rate $R = \frac{k}{n}$ Constraint Length $= n(m+1)$

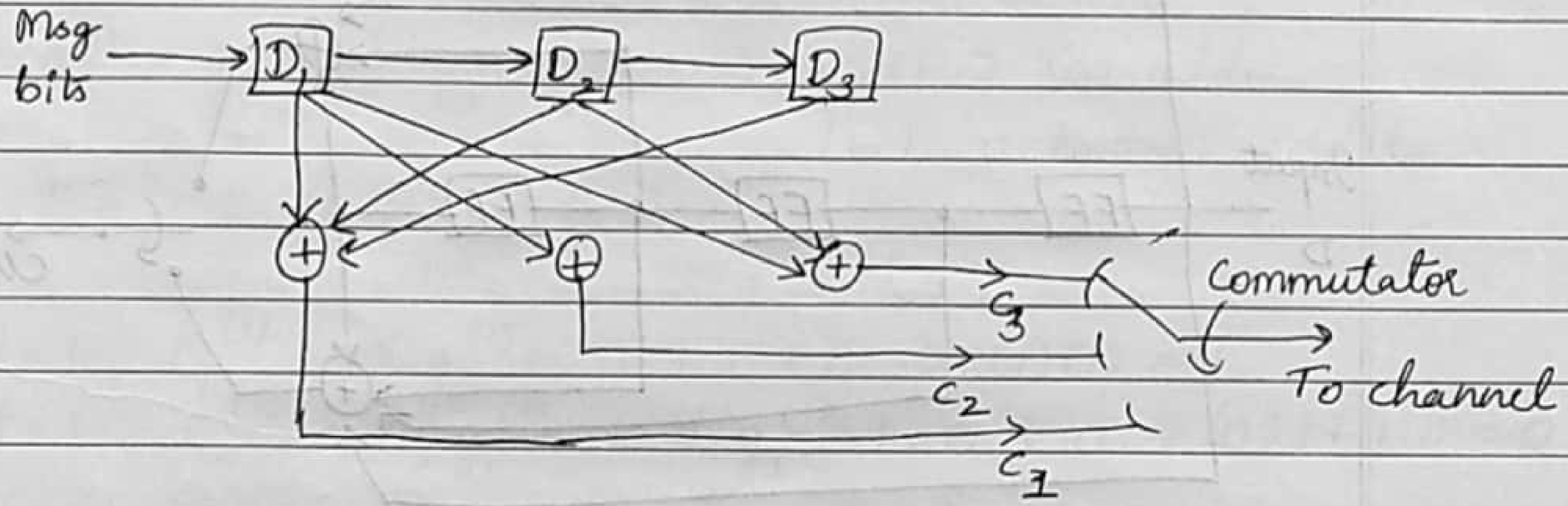
28/10

Module-5 - Part B

Convolutional Codes

It is well suited for error correction.

Encoder for $(n, k, m) = (3, 1, 3)$ convolutional code is shown in figure below.



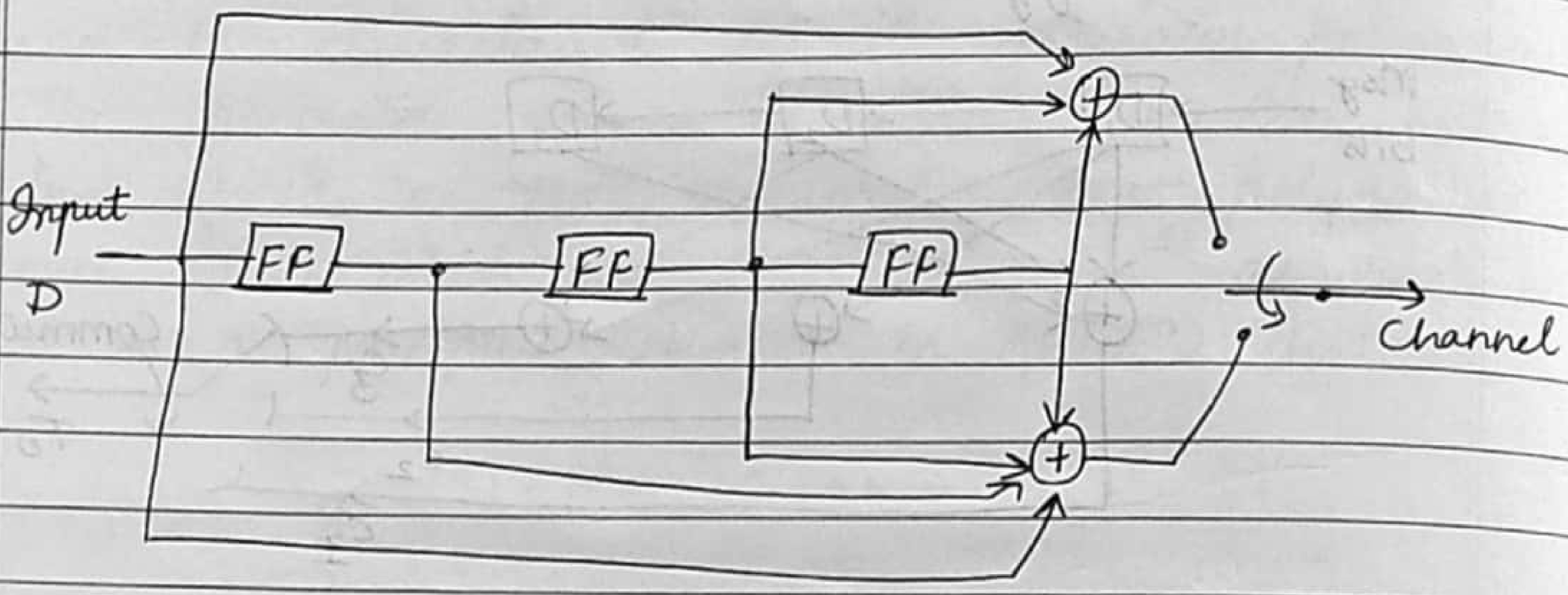
$n \rightarrow$ Number of outputs $\Rightarrow$ No. of modulo 2 adders
$k \rightarrow$ No. of input bits entered at any time
$m \rightarrow$ No. of FF's in shift registers

Let us take the message as 10110 and analyze the circuit

0	T_b	$2T_b$	$3T_b$	$4T_b$	$5T_b$	$6T_b$	$7T_b$	
1	0	1	1	0	0	0		← I/p
1 0 0	0 1 0	1 0 1	1 1 0	0 1 1	0 1 1	0 0 0		msg
1 1 1	1 0 1	0 1 1	0 1 0	0 0 1	1 0 0	0 0 0		← o/p

Time Domain Approach

Let us consider $(n, k, m) \rightarrow (2, 1, 3)$ convolutional encoder as shown in figure below



⇒ defined by $n \rightarrow$ impulse response

Rate efficiency is defined by k/n

Length of code = $(L+M)$
$L \rightarrow$ no. of bits in the message

$C^{(1)}$, $C^{(2)}$ } impulse responses coded of the adders

$$C^{(1)} = [d] * g^{(1)}$$
$$C^{(2)} = [d] * g^{(2)}$$

$$C_l^{(i)} = \sum_{i=0}^{m} d_{l-i} \, g_{i+1}^{(j)}$$

Let message signal be $d_1 d_2 d_3 d_4 d_5 = 10111$

for $j = 1$

$$C_l^{(1)} = \sum_{i=0}^{3} d_{l-i}\, g_{i+1}^{(1)} \quad \text{where } d_{l-i} = 0 \text{ for } \forall\ l \leq i$$

$$C_l^{(1)} = d_l g_1^{(1)} + d_{l-1} g_2^{(1)} + d_{l-2} g_3^{(1)} + d_{l-3} g_4^{(1)}$$

$g^{(1)} = g_1^{(1)} g_2^{(1)} g_3^{(1)} g_4^{(1)} \Rightarrow 1011 \rightarrow$ Top adder

$1111 \rightarrow$ Bottom adder

$l \rightarrow 1$ to $(L+M) = 1$ to 8

$l = 1,\ C_1^{(1)} = d_1 g_1^{(1)} + 0 + 0 + 0 \Rightarrow (1)(1) = 1$

$l = 2,\ C_2^{(1)} = d_2 g_1^{(1)} + d_1 g_2^{(1)} + 0 + 0 \Rightarrow (0.1) + 1.0 = 0$

$l = 3;\ C_3^{(1)} = d_3 g_1^{(1)} + d_2 g_2^{(1)} + d_1 g_3^{(1)} + 0 \Rightarrow 1.1 + 0.0 + 1.1$

$= 0$

$l = 4,\ C_4^{(1)} = d_4 g_1^{(1)} + d_3 g_2^{(1)} + d_2 g_3^{(1)} + d_1 g_4^{(1)}$

$= 1.1 + 1.0 + 0.1 + 1.1 = 1 + 0 + 0 + 1 = 0$

$l = 5,\ C_5^{(1)} = d_5 g_1^{(1)} + d_4 g_2^{(1)} + d_3 g_3^{(1)} + d_2 g_4^{(1)}$

$= (1.1) + (1.0) + (1.1) + (0.1)$

$= 1 + 0 + 1 + 0 = 0$

$l = 6,\ C_6^{(1)} = d_6 g_1^{(1)} + d_5 g_2^{(1)} + d_4 g_3^{(1)} + d_3 g_4^{(1)}$

$= (0.1) + (1.0) + (1.1) + (1.1) = 1 + 1 = 0$

$l = 7,\ C_7^{(1)} = d_7 g_1^{(1)} + d_6 g_2^{(1)} + d_5 g_3^{(1)} + d_4 g_4^{(1)}$

$= (0.1) + (0.0) + (1.1) + (1.1)$

$= 0$

$l = 8,\ C_8^{(1)} = d_8 g_1^{(1)} + d_7 g_2^{(1)} + d_6 g_3^{(1)} + d_5 g_4^{(1)}$

$= (0.1) + (0.0) + (0.1) + (1.1)$

$= 1$

31/10

$g^{(1)} = g_1^{(1)} g_2^{(1)} g_3^{(1)} g_4^{(1)} = 1011$

$g^{(2)} = g_1^{(2)} g_2^{(2)} g_3^{(2)} g_4^{(2)} = 1111$

d= 10111

l varies from 1 to (L+M)

1 to (5+3) = 1 to 8

l=1 $C_1^{(1)} = d_L g_1^{(1)} + d_{l-1} g_1^{(2)} + d_{l-2} g_1^{(3)} + d_{l-3} g_1^{(4)}$

$d_{l-i} = 0, \forall\ l \le i$ $\quad C_1^{(1)} = d_1 g_1^{(1)} + 0+0+0$

$= (1)(1) = 1$

$C_2^{(1)} = d_2 g_1^{(1)} + d_1 g_2^{(1)} + 0+0 = 0(1)+(1)(0) = 0$

$C_3^{(1)} = d_3 g_1^{(1)} + d_2 g_2^{(1)} + d_1 g_3^{(1)} + 0 = (1)(1)+(0)0+1(1)$

$= 0$

$C_4^{(1)} = 0, \quad C_5^{(1)} = 0, \quad C_6^{(1)} = 0 \quad C_7^{(1)} = 0, \quad C_8^{(1)} = 1$

$C^{(1)} = 10000001$

$C_1^{(2)} = 1$

$C_2^{(2)} = 1$

$C_1^{(2)} = 11011101$

$C = [C_1^{(1)} C_1^{(2)}, C_2^{(1)} C_2^{(2)} \cdots C_{L+m}^{(1)} \; C_{L+m}^{(2)}]$

$C = [11, 01, 00, 01, 01, 01, 00, 11]$

Matrix Method

$L \times n(L+m)$ $\quad L \to$ no. of bits in msg sequence

$L = 5$ $\quad n \to$ no. of adders

$n = 2$ $\quad m \to$ no. of shift registers

$m = 3$ $\quad (5 \times 16)$

31/10

top adder ↑ bottom adder ↑

$g_{m+1} = g_4 = 8$ col.
other 8 col = 0

$$[G] = \begin{bmatrix} g_1^{(1)} & g_2^{(2)} & g_2^{(1)} & g_2^{(2)} & g_3^{(1)} & g_3^{(2)} & \cdots & g_{m+1}^{(1)} & g_{m+1}^{(2)} & 0 0 \cdots 0 0 \\ 0 & 0 & g_1^{(1)} & g_1^{(2)} & g_2^{(1)} & g_2^{(2)} & \cdots & \cdots & g_{m+1}^{(1)} g_{m+1}^{(2)} & \cdots 00 \\ 0 & 0 & 0 & 0 & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \; 0 & g_1^{(1)} \; g_1^{(2)} & \cdots \; g_{m+1}^{(1)} \; g_{m+1}^{(2)} \end{bmatrix}$$

2nd row start with 0's equal to no. of adders and shift the I row towards right by 2

$g^{(1)} = 1011 \qquad g^{(2)} = 1111$

$$[G] = \begin{bmatrix} 11 & 01 & 11 & 11 & 0\,0 & 0\,0 & 00 & 00 \\ 00 & 11 & 01 & 11 & 1\,1 & 00 & 00 & 00 \\ 00 & 00 & 11 & 01 & 1\,1 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 01 & 11 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 01 & 11 & 11 \end{bmatrix}$$

$$C = [D][G]$$

$$= [10111]_{1\times 5} \begin{bmatrix} 11 & 01 & 11 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 01 & 11 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 01 & 11 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 01 & 11 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 01 & 11 & 11 \end{bmatrix}_{5\times 16}$$

$$= [1101000101010011]$$

$$= [11, 01, 00, 01, 01, 01, 00, 11]$$

1/10/19

Transform Domain Approach

The generator impulse response is represented as

$$g^{j}(x) = g_1^{(j)} + g_2^{(j)}x + g_3^{(j)}x^2 + g_4^{(j)}x^3 + \dots + g_{m+1}^{(j)}x^m \quad ①$$

j varies from 1 to n

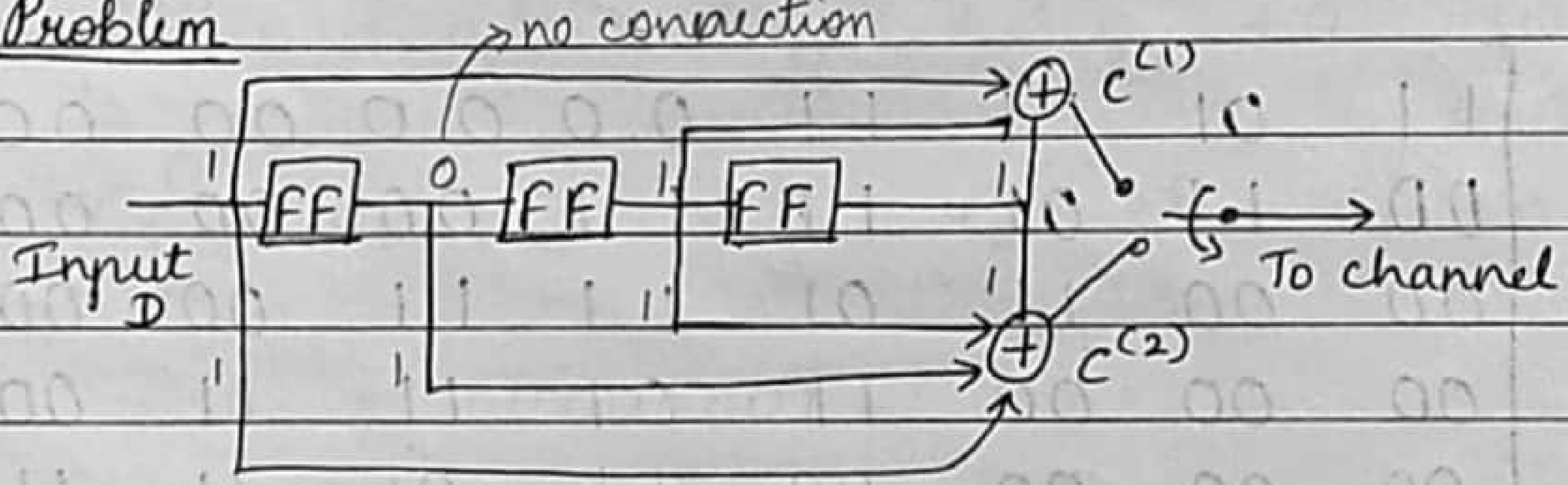
$n \rightarrow$ no. of adders

$$C^{(j)}x = d(x)\, g^{(j)}(x) \quad ②$$ j varies from 1 to n

$$c(x) = C^{(1)}(x^n) + xC^{(2)}(x^n) + x^2 C^{(3)}(x^n) + \dots + x^{n-1}C^{(n)}(x^{n}$$ ③

Problem

Ⓠ



$g^{(1)} = 1011 \qquad g^{(2)} = 1111$

$g^{(1)}(x) = 1 + 0 + x^2 + x^3 = 1 + x^2 + x^3$

$g^{(2)}(x) = 1 + x + x^2 + x^3$

$d = 10111 \qquad d(x) = 1 + x^2 + x^3 + x^4$

$$C^{(1)}(x) = d(x)\, g^{(1)}(x) = (1+x^2+x^3+x^4)(1+x^2+x^3)$$

$$= 1 + \cancel{x^2} + \cancel{x^3} + \cancel{x^2} + \cancel{x^4} + \cancel{x^5} + \cancel{x^3} + \cancel{x^5} + \cancel{x^6} + \cancel{x^4} + \cancel{x^6} + x^7$$

$$= 1 + x^7$$

$$c^{(2)}(x) = d(x)\,g^{(2)}(x) = (1+x^2+x^3+x^4)(1+x^2+x+x^3)$$

$$= 1+\cancel{x^2}+x+\cancel{x^3}+\cancel{x^2}+x^4+\cancel{x^3}+\cancel{x^5}+x^3+\cancel{x^5}+\cancel{x^4}+\cancel{x^6}+\cancel{x^4}+\cancel{x^6}+x^5+x^7$$

$$= 1+x+x^3+x^4+x^5+x^7$$

~~$c(x) = c^{(1)}(x^2) + x\,c^{(2)}(x^2)$~~

~~$= (1+x^{7})x^2 + x^3(1+x+x^3+x^4+x^5+x^7)$~~

~~$= x^2+x^9 + x^3+x^4+x^6+x^7+x^8+x^{10}$~~

~~$c(x) = x^2+x^3+x^4+x^6+x^7+x^8+x^9+x^{10}$~~

$$c(x) = c^{(1)}(x^2) + x\,c^{(2)}(x^2)$$

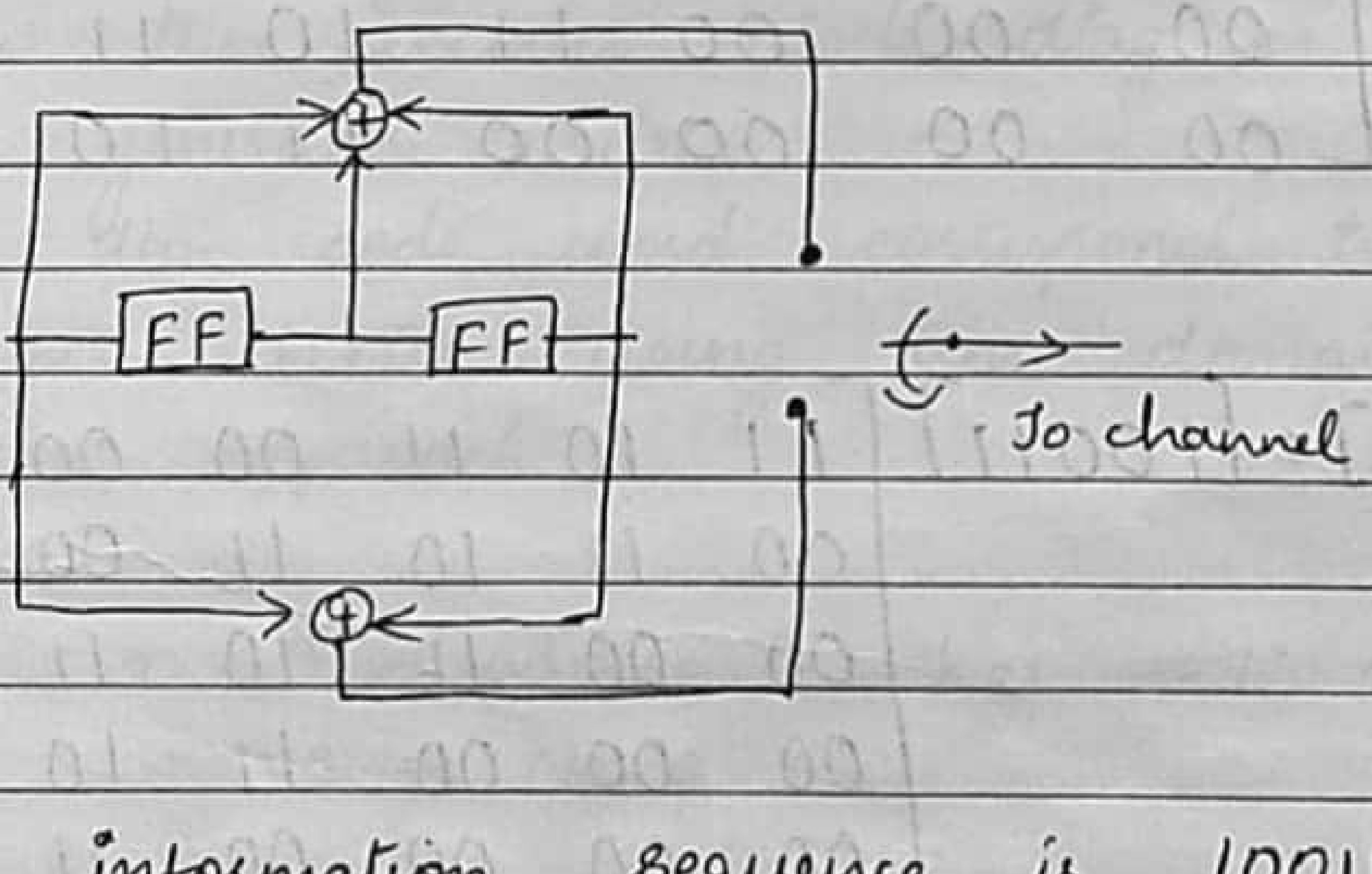
$$c^{(1)}(x^2) = 1+x^{14} \qquad c^{(2)}(x^2) = 1+x^2+x^6+x^8+x^{10}+x^{14}$$

$$c(x) = 1+x^{14} + x+x^3+x^7+x^9+x^{11}+x^{15}$$

$$c(x) = 1+x+x^3+x^7+x^9+x^{11}+x^{14}+x^{15}$$

$$[c] = [11,\ 01,\ 00,\ 01,\ 01,\ 01,\ 00,\ 11]$$

②



The information sequence is 10011. Find the output sequence using

a) time domain approach b) Transform domain approach

Sol. a) $g^{(1)}(x) = 111 = g^{(1)}$

$g^{(2)}(x) = 101 = g^{(2)}$

$d = 10011$

$L \times n(m+L)$

$L = 5$

$n = 2$

$m = 2$

5×14

$$[G] = \begin{bmatrix} g_1^{(1)} & g_1^{(2)} & g_2^{(1)} & g_2^{(2)} & \cdots & g_{m+1}^{(1)} & g_{m+1}^{(2)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & g_1^{(1)} & g_1^{(2)} & & & - - - - & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & g_1^{(1)} & g_1^{(2)} & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & g_1^{(1)} & g_1^{(2)} & \cdots & g_{m+1}^{(2)} \end{bmatrix}$$

$$[G] = \begin{bmatrix} 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 10 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 10 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 10 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 10 & 11 \end{bmatrix}$$

$$C = [D][G] = [10011] \begin{bmatrix} 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 10 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 10 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 10 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 10 & 11 \end{bmatrix}$$

$= [11, 10, 11, 11, 01, 01, 11]$

b) $g^{(1)} = 111$ $\quad g^{(2)} = 101$

$g^{(1)}(x) = 1 + x + x^2$ $\quad g^{(2)}(x) = 1 + x^2$ $\quad c^{(j)}(x) = d(x)g^{(j)}(x)$

$d = 10011$ $\quad d(x) = 1 + x^3 + x^4$

$$c^{(1)}(x) = d(x)g^{(1)}(x) = (1+x^3+x^4)(1+x+x^2)$$
$$= 1 + x^3 + x^4 + x + x^4 + x^5 + x^2 + x^5 + x^6$$
$$= 1 + x + x^2 + x^3 + x^6$$

$$c^{(2)}(x) = d(x)g^{(2)}(x) = (1+x^3+x^4)(1+x^2) = 1 + x^5 + x^6 + x^2 + x^3 + x^4$$
$$= 1 + x^2 + x^3 + x^4 + x^5 + x^6$$

$$c(x) = c^{(1)}(x^2) + xc^{(2)}(x^2)$$
$$= 1 + x + x^4 + x^6 + x^{12} + x[1 + x^4 + x^6 + x^8 + x^{10} + x^{12}]$$
$$= 1 + x^2 + x^4 + x^6 + x^{12} + x + x^5 + x^7 + x^9 + x^{11} + x^{13}$$
$$= 1 + x + x^2 + x^4 + x^6 + x^7 + x^9 + x^5 + x^{11} + x^{12} + x^{13} = 1 + x + x^2 + x^4 + x^5 + x^6 + x^7 + x^9 + x^{11} + x^{12} + x^{13}$$

$c = [11, 10, 11, 11, 01, 01, 11]$

③ Consider (3,12) convolutional code with $g^{(1)} = 110$ $g^{(2)} = 101$ $g^{(3)} = 111$

i) Draw the encoder block diagram

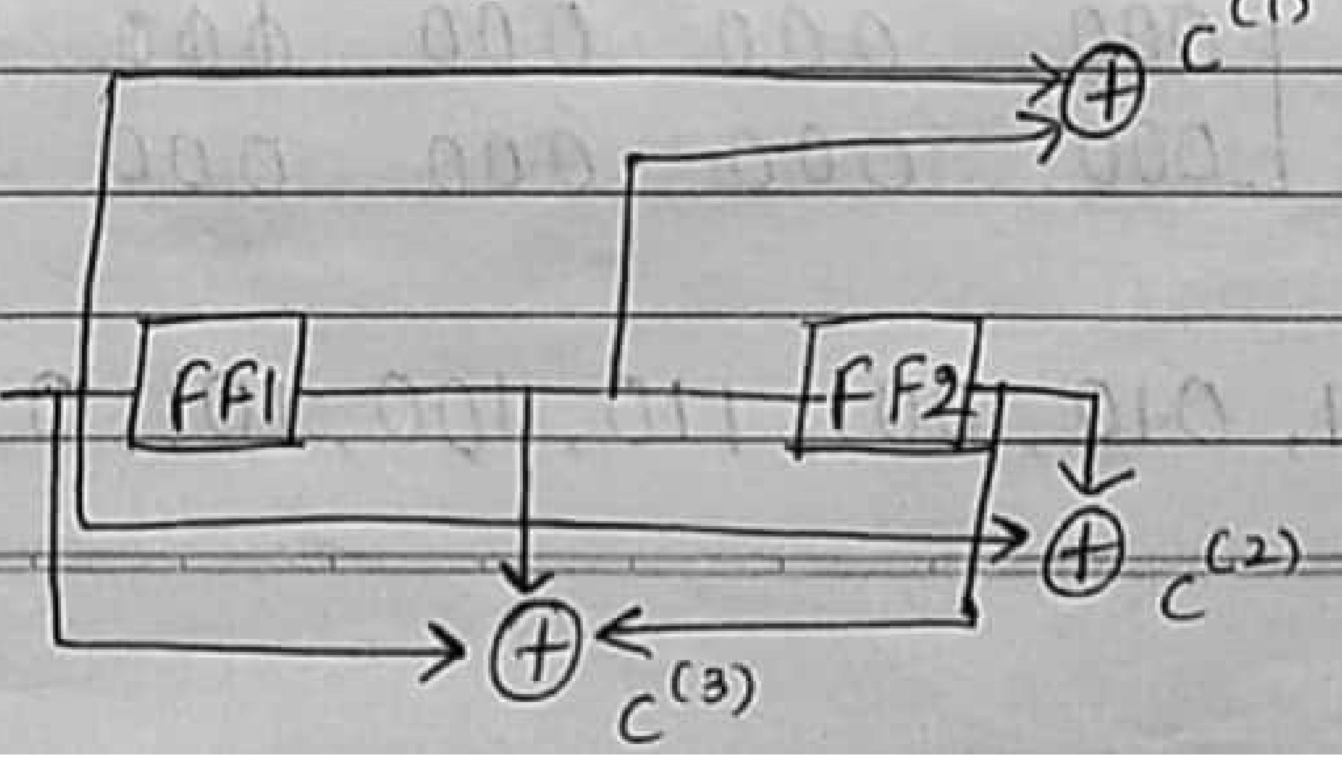
ii) Find generator matrix.

iii) Find the code word correspond to the information sequence 11101 using time domain and transfer domain approach.

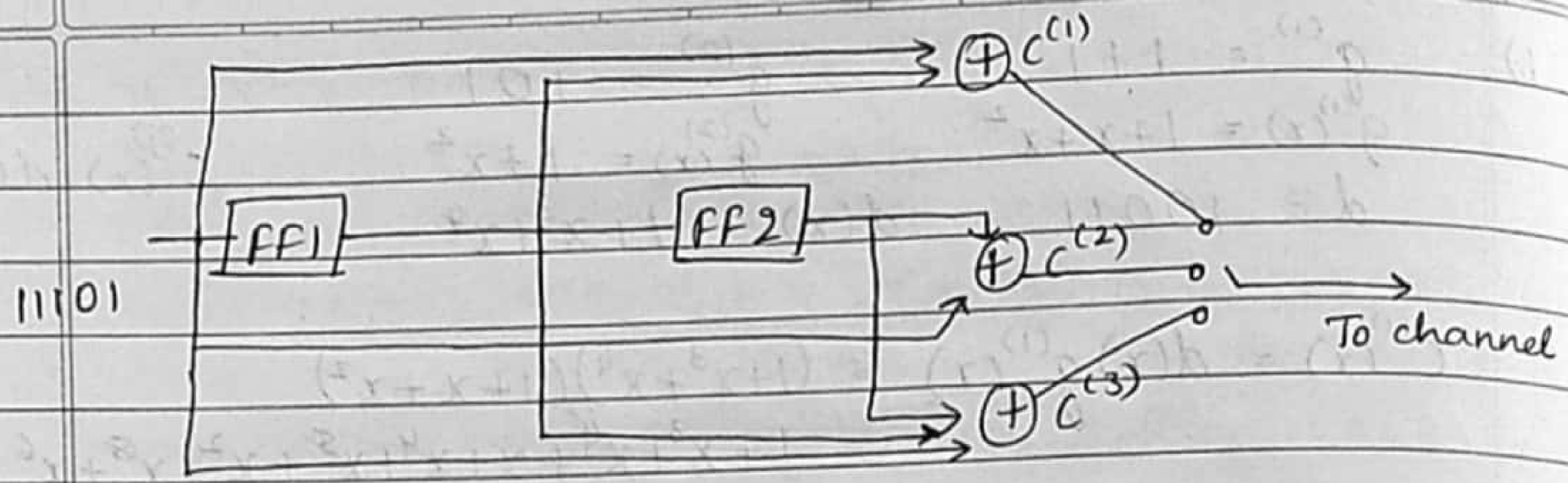
Sol g $\quad n = 3 \Rightarrow$ adders $\quad k = 1 \Rightarrow$ No. of bits at a time

$m = 2 \Rightarrow$ No. of ff

i) g



OR



ii) $g^{(1)} = 110 \quad g^{(2)} = 101 \quad g^{(3)} = 111$

$L \times n(L+M) =$

$L = 5 \Rightarrow$ Length of msg sequence

$5 \times 3(5+2) \Rightarrow 5 \times 21$

$$[G] = \begin{bmatrix} 1 & 1 & 1 & 1\,0\,1 & 0\,1\,1 & 0\,0\,0 & 000 & 000 & 000 \\ 0 & 0 & 0 & 1\,1\,1 & 1\,0\,1 & 0\,1\,1 & 000 & 000 & 000 \\ 0 & 0 & 0 & 0\,0\,0 & 1\,1\,1 & 1\,0\,1 & 011 & 000 & 000 \\ 0 & 0 & 0 & 0\,0\,0 & 000 & 1\,1\,1 & 101 & 011 & 000 \\ 0 & 0 & 0 & 0\,0\,0 & 000 & 000 & 111 & 101 & 011 \end{bmatrix}$$

iii) Time domain approach

$$C = [D][G]$$

$$= [11101] \begin{bmatrix} 111 & 101 & 011 & 000 & 000 & 000 & 000 \\ 000 & 111 & 101 & 011 & 000 & 000 & 000 \\ 000 & 000 & 111 & 101 & 011 & 000 & 000 \\ 000 & 000 & 000 & 000 & 101 & 011 & 000 \\ 000 & 000 & 000 & 000 & 111 & 101 & 011 \end{bmatrix}$$

$$= [111, 010, 001, 110, 100, 101, 011]$$

Transfer domain approach

$g^{(1)} = 110 \quad g^{(2)} = 101 \quad g^{(3)} = 111$

$g^{(1)}(x) = 1+x \quad g^{(2)}(x) = 1+x^2 \quad g^{(3)}(x) = 1+x+x^2$

$d = 11101 \quad d(x) = 1+x+x^2+x^4$

$$c^{(1)}(x) = d(x)g^{(1)}(x) = (1+x+x^2+x^4)(1+x)$$
$$= 1+\cancel{x}+\cancel{x}+\cancel{x^2}+\cancel{x^2}+x^3+x^4+x^5$$
$$= 1+x^3+x^4+x^5$$

$$c^{(2)}(x) = d(x)g^{(2)}(x) = (1+x+x^2+x^4)(1+x^2)$$
$$= 1+\cancel{x^2}+x+x^3+\cancel{x^2}+\cancel{x^4}+\cancel{x^4}+x^6$$
$$= 1+x+x^3+x^6$$

$$c^{(3)}(x) = d(x)g^{(3)}(x) = (1+x+x^2+x^4)(1+x+x^2)$$
$$=1+\cancel{x}+\cancel{x^2}+\cancel{x}+\cancel{x^2}+\cancel{x^3}+x^2+\cancel{x^3}+\cancel{x^4}+\cancel{x^4}+x^5+x^6$$
$$= 1+x^2+x^5+x^6$$

$$c(x) = c^{(1)}(x^3) + x\,c^{(2)}(x^3) + x^2 c^{(3)}(x^3)$$

$$c^{(1)}(x^3) = 1+x^9+x^{12}+x^{15}$$
$$c^{(2)}(x^3) = 1+x^3+x^9+x^{18}$$
$$c^{(3)}(x^3) = 1+x^6+x^{15}+x^{18}$$

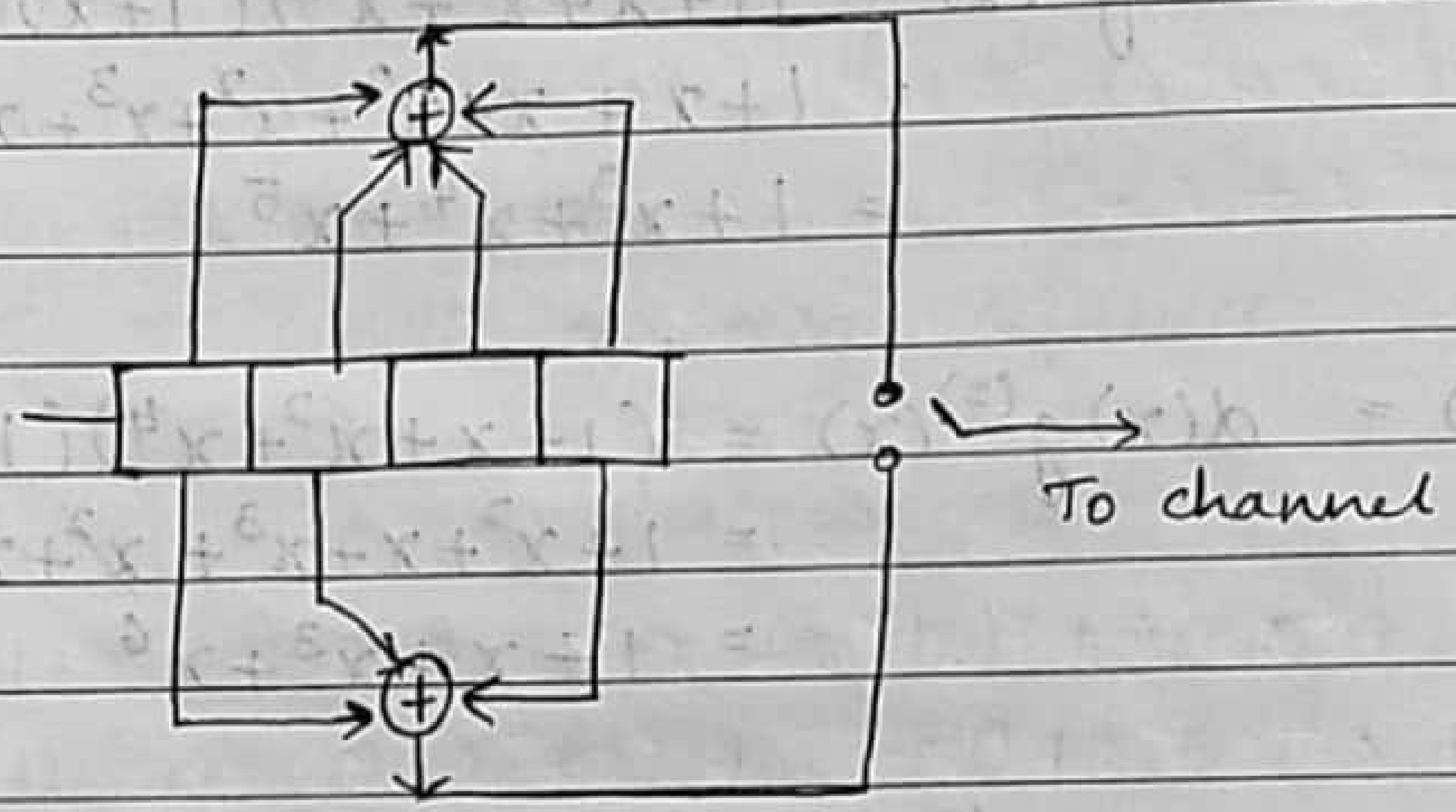
$$c(x) = 1+x^9+x^{12}+x^{15}+x+x^4+x^{10}+x^{19}+x^2+x^8+x^{17}+x^{20}$$
$$= 1+x+x^2+x^4+x^8+x^9+x^{10}+x^{12}+x^{15}+x^{17}+x^{19}+x^{20}$$

$c = [111, 010, 001, 110, 100, 101, 011]$

④ For the convolutional encoder shown in figure below

i) Find the impulse response and hence calculate the output produced by info. sequence 10111.

ii) Write the generator polynomials of the encoder and recompute the output for input of 10111 and compare it with that of (i).



Sol^n:- When impulse response is mentioned then use time domain 1st approach.

For ii) part → use transfer domain approach → polynomials

i) $g^{(1)} = 01111$ $g^{(2)} = 01101$ $d = 10111$

$L = 5$

$M = 4$

$$C_l^{(i)} = \sum_{i=1}^{m} [d]_{l-i}\, g_{i+1}^{(j)}$$

$l \to 1$ to $(L+M) = 1$ to 9

$j = 1$

$C_1^{(1)} = d_1 g_1^{(1)} * \cancel{d_2} + 0 + 0 = 0$

$C_2^{(1)} = d_2 g_1^{(1)} + d_1 g_2^{(1)} = 0(0) + 1(1) = 1$

$C_3^{(1)} = d_3 g_1^{(1)} + d_2 g_2^{(1)} + d_1 g_3^{(1)} = 1(0) + 0(1) + 1(1) = 01$

$C_4^{(1)} = d_4 g_1^{(1)} + d_3 g_2^{(1)} + d_2 g_3^{(1)} + d_1 g_4^{(1)} = 1(0) + 1(1) + 0(1) + 1(1)$

$= 0$

$C_5^{(1)} = d_5 g_1^{(1)} + d_4 g_2^{(1)} + d_3 g_3^{(1)} + d_2 g_4^{(1)} + d_1 g_5^{(1)}$
$= 1(0) + 1(1) + 1(1) + 0(1) + 1(1) = 1$

$C_6^{(1)} = d_6 g_1^{(1)} + d_5 g_2^{(1)} + d_4 g_3^{(1)} + d_3 g_4^{(1)} + d_2 g_5^{(1)}$
$= 0(0) + 1(1) + 1(1) + 1(1) + 0(1) = 1$

$C_7^{(1)} = d_7 g_1^{(1)} + d_6 g_2^{(1)} + d_5 g_3^{(1)} + d_4 g_4^{(1)} + d_3 g_5^{(1)}$
$= 0(0) + 0(1) + 1(1) + 1(1) + 1(1) = 1$

$C_8^{(1)} = d_8 g_1^{(1)} + d_7 g_2^{(1)} + d_6 g_3^{(1)} + d_5 g_4^{(1)} + d_4 g_5^{(1)}$
$= 0(0) + 0(1) + 0(1) + 1(1) + 1(1) = 0$

$C_9^{(1)} = d_8 g_1^{(1)} + d_8 g_2^{(1)} + d_7 g_3^{(1)} + d_6 g_4^{(1)} + d_5 g_5^{(1)} = 1$

$j = 2.$

$C_1^{(2)} = d_1 g_1^{(2)} = 1(0) = 0$

$C_2^{(2)} = d_2 g_1^{(2)} + d_1 g_2^{(2)} = 0(0) + 1(1) = 1$

$C_3^{(2)} = d_3 g_1^{(2)} + d_2 g_2^{(2)} + d_1 g_3^{(2)} = 1(0) + 0(1) + 1(1) = 1$

$C_4^{(2)} = d_4 g_1^{(2)} + d_3 g_2^{(2)} + d_2 g_3^{(2)} + d_1 g_4^{(2)}$
$= 1(0) + 1(1) + 0(1) + 1(0) = 1$

$C_5^{(2)} = d_5 g_1^{(2)} + d_4 g_2^{(2)} + d_3 g_3^{(2)} + d_2 g_4^{(2)} + d_1 g_5^{(2)}$
$= 1(0) + 1(1) + 1(1) + 0(0) + 1(1) = 1$

$C_6^{(2)} = d_6 g_1^{(2)} + d_5 g_2 + d_4 g_3 + d_3 g_4 + d_2 g_5$
$= 0(0) + 1(1) + 1(1) + 1(0) + 0(1) = 0$

$C_7^{(2)} = d_7 g_1 + d_6 g_2 + d_5 g_3 + d_4 g_4 + d_3 g_5$
$= 0(0) + 0(1) + 1(1) + 1(0) + 1(1) = 0$

$C_8^{(2)} = d_8 g_1 + d_7 g_2 + d_6 g_3 + d_5 g_4 + d_4 g_5$
$= 0(0) + 0(1) + 0(1) + 1(0) + 1(1) = 1$

$C_9^{(2)} = d_9 g_1 + d_8 g_2 + d_7 g_3 + d_6 g_4 + d_5 g_5$
$= 0(0) + 0(1) + 0(1) + 0(0) + 1(1) = 1$

$C = [00, 11, 11, 01, 11, 10, 10, 01, 11]$

Scanned with CamScanner

ii) $g^{(1)} = 01111$ $g^{(2)} = 01101$

$g^{(1)}(x) = x+x^2+x^3+x^4$; $g^{(2)}(x) = x+x^2+x^4$

$g^{j}(x) = g_1^{(j)} + g_2^{(j)}x + g_3^{(j)}x^2 + g_4^{(j)}x^3 + \dots + g_{m+1}^{(j)}x^m$

$d(x) = 1+x^2+x^3+x^4$

$$\begin{aligned} c^{(1)}(x) &= d(x)\,g^{(1)}(x) \\ &= (1+x^2+x^3+x^4)(x+x^2+x^3+x^4) \\ &= x+x^2+\cancel{x^3}+\cancel{x^4}+\cancel{x^3}+\cancel{x^4}+\cancel{x^6}+\cancel{x^5}+x^4+\cancel{x^5}+\cancel{x^6}+\cancel{x^7} \\ &\quad +x^5+x^6+\cancel{x^7}+x^8 \\ &= x+x^2+x^4+x^5+x^6+x^8 \end{aligned}$$

$$\begin{aligned} c^{(2)}(x) &= d(x)\,g^{(2)}(x) \\ &= (1+x^2+x^3+x^4)(x+x^2+x^4). \\ &= x+x^2+\cancel{x^4}+\cancel{x^3}+\cancel{x^4}+\cancel{x^6}+\cancel{x^4}+\cancel{x^5}+x^7+\cancel{x^8}+\cancel{x^6}+x^8 \\ &= x+x^2+x^7+x^8+x^3+x^4 \end{aligned}$$

$c(x) = c^{(1)}(x^2) + x\,c^{(2)}(x^2)$

$c^{(1)}(x^2) = x^2+x^4+x^8+x^{10}+x^{12}+x^{16}$

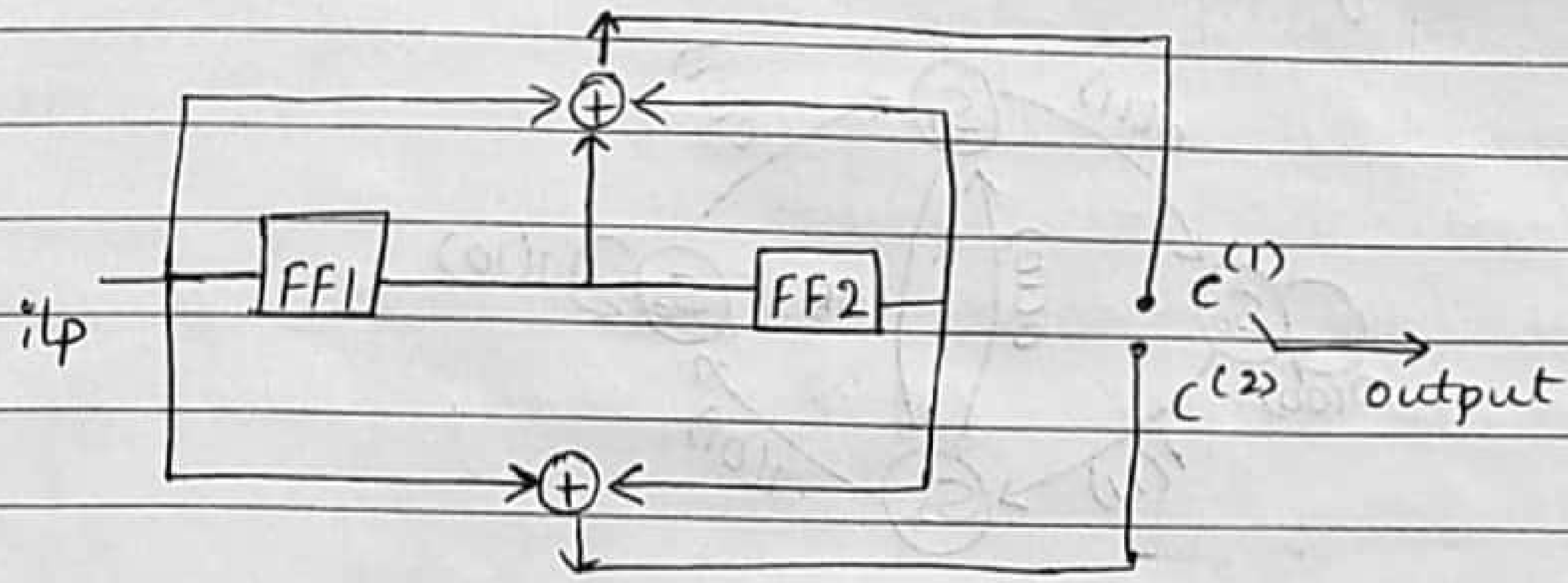
$c^{(2)}(x^2) = x^2+x^4+x^6+x^8+x^{14}+x^{16}$

~~$c(x) = \cancel{x^2}+\cancel{x^4}+\cancel{x^8}+x^{10}+x^{12}+\cancel{x^{16}}+\cancel{x^2}+\cancel{x^4}+x^6+\cancel{x^8}+x^{14}+x^{1}$~~

$$\begin{aligned} c(x) &= x^2+x^4+x^8+x^{10}+x^{12}+x^{16}+x^3+x^5+x^7+x^9+x^{15}+x^{17} \\ &= x^2+x^3+x^4+x^5+x^7+x^8+x^9+x^{10}+x^{12}+x^{15}+x^{16}+x^{17} \end{aligned}$$

$c = [00, 11, 11, 01, 11, 10, 10, 01, 11]$

State diagrams and code tree



(n, k, m)

(2, 1, 2)

$$c^{(1)} = d_l + d_{l-1} + d_{l-2}$$
$$c^{(2)} = d_l + d_{l-2}$$

Each FF can store 1 bit

There are 2 FF and 2 bits can be stored.

So there will be 2^2 states

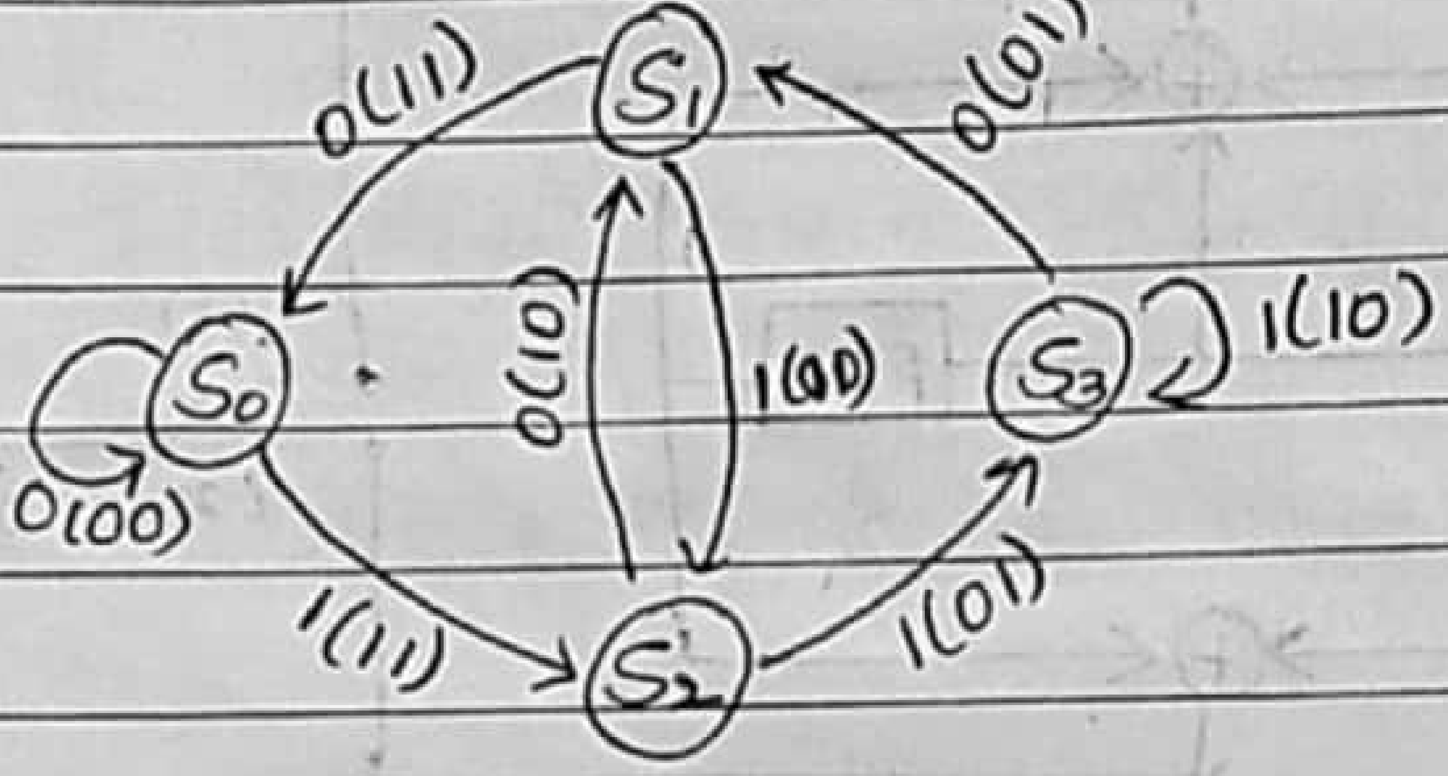
State	Binary description
S_0	00
S_1	01
S_2	10
S_3	11

State table

State transition table.

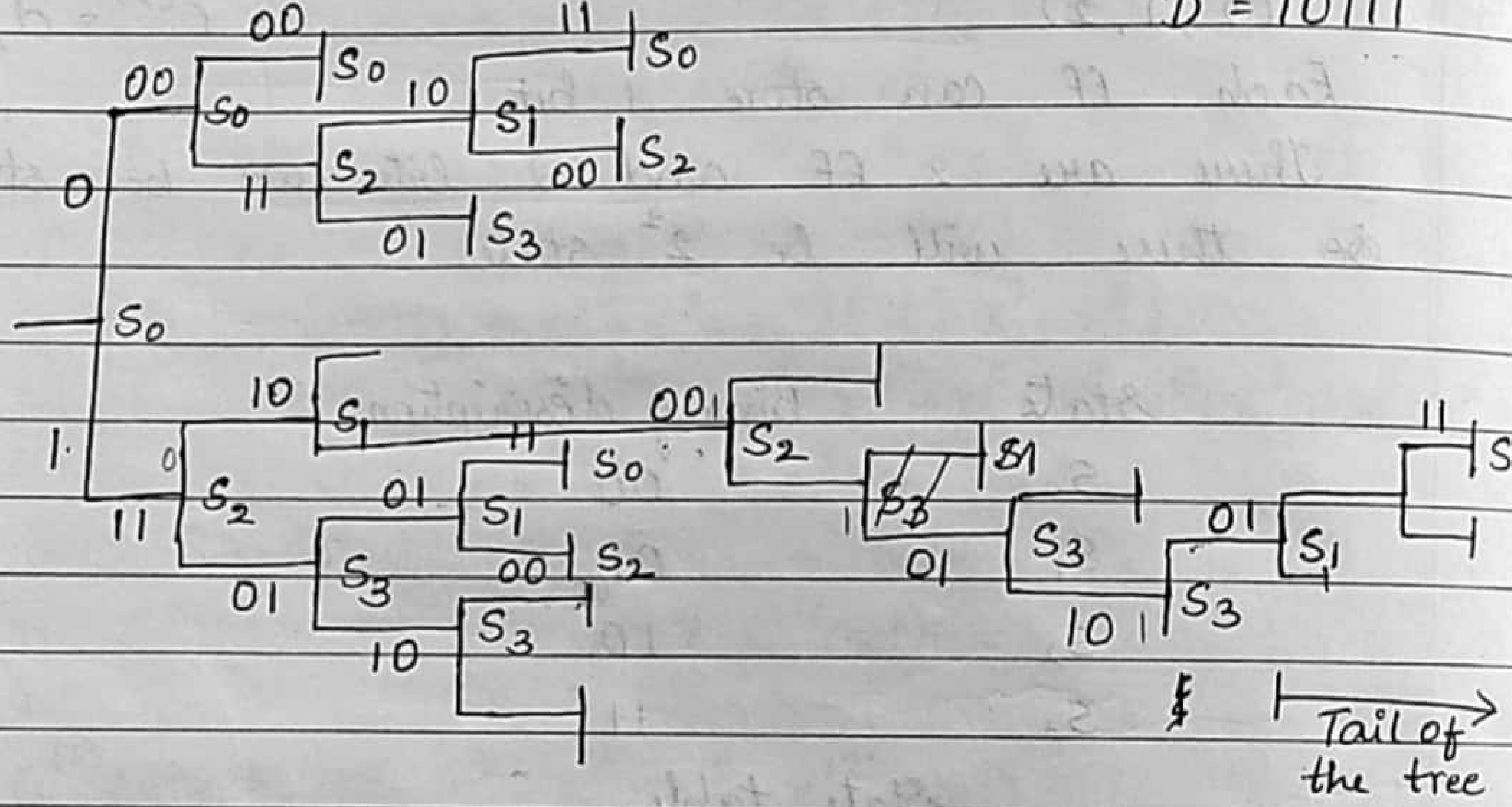
Present state	Binary description	Input	Next state	Binary description	d_l	d_{l-1}	d_{l-2}	Output $c^{(1)}$	$c^{(2)}$
S_0	00	0	S_0	0 0	0	0	0	0	0
		1	S_2	1 0	1	0	0	1	1
S_1	01	0	S_0	0 0	0	0	1	1	1
		1	S_2	1 0	1	0	1	0	0
S_2	10	0	S_1	0 1	0	1	0	1	0
		1	S_3	1 1	1	1	0	0	1
S_3	11	0	S_1	0 1	0	1	1	0	1
		1	S_3	1 1	1	1	1	1	0

State diagram



Code tree

Consider
D = 10111



	FF1	FF2
	0	0
1 →	1	0
0 →	0	1
1 →	1	0
1 →	1	1
1 →	1	1
0 →	0	1
0 →	0	0

We have 2 FF so apply 2 clk pulse to take input bit outside the ckt

If we have n FF then apply n clk pulses.

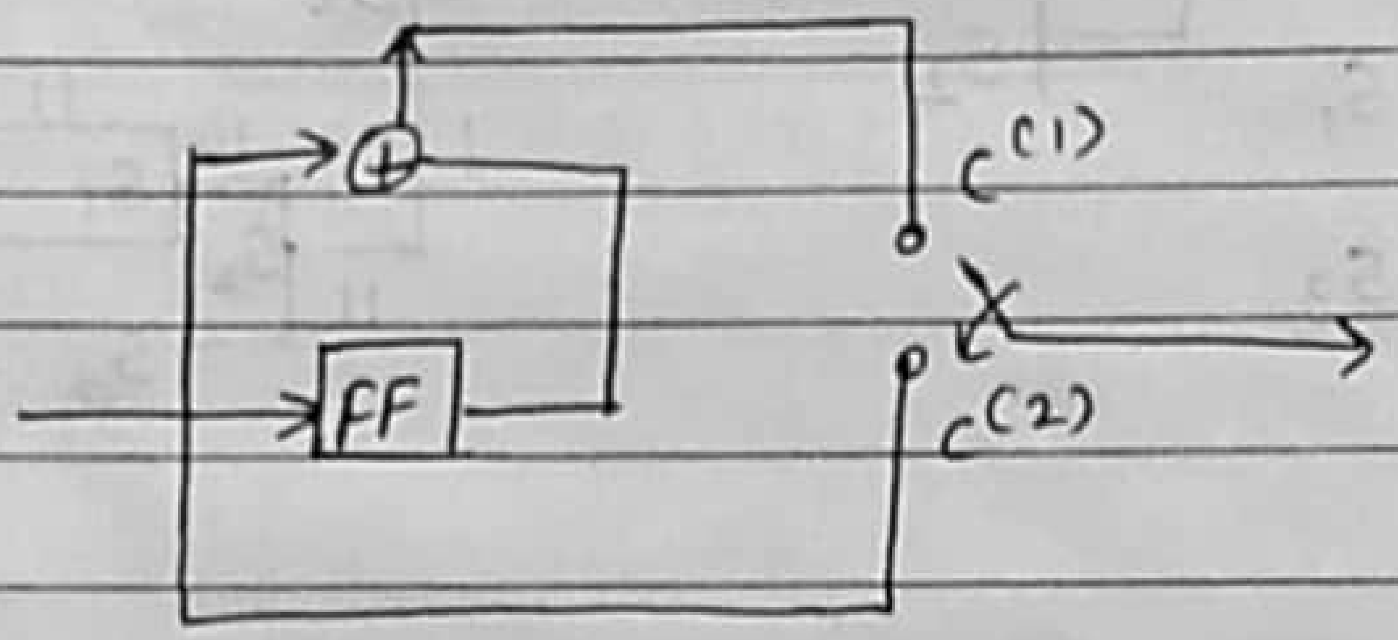
C = [11, 10, 00, 01, 10, 01, 11]

Ⓠ Consider the convolutional encoder shown in figure. The code is systematic.

i) Draw the state diagram

ii) Draw the code tree

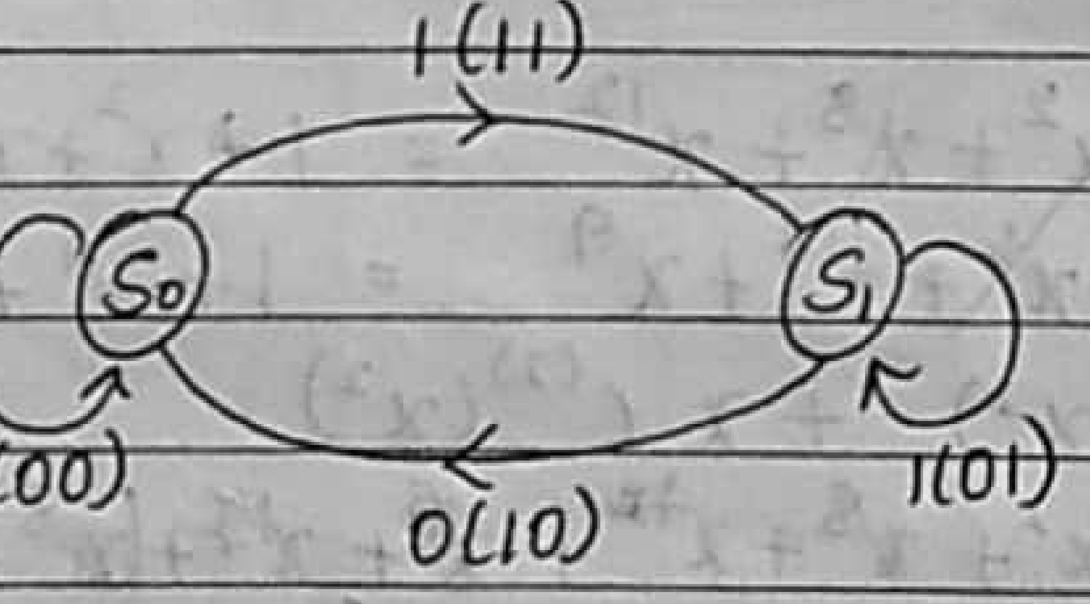
iii) Find the encoder output produced using sequence 10111 verify using time domain approa

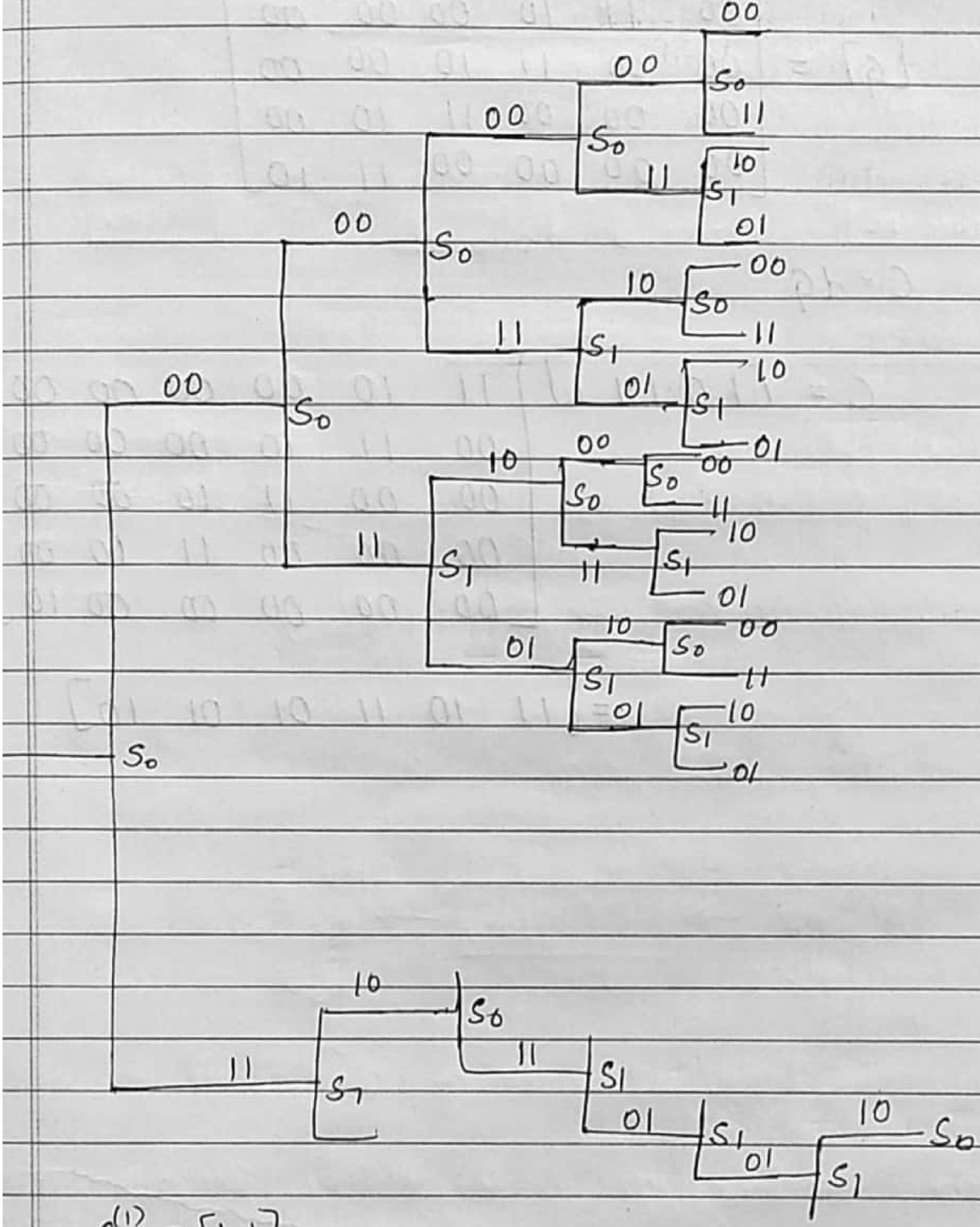


$$c^{(1)} = d_l + d_{l-1}$$
$$c^{(2)} = d_l$$

Present state	Binary description	Input	Next state	Binary description	d_l	d_{l-1}	Output $c^{(1)}$	$c^{(2)}$
S_0	0	0	S_0	0	0	0	0	0
		1	S_1	1	1	0	1	1
S_1	1	0	S_0	0	0	1	1	0
		1	S_1	1	1	1	0	1

i) State Diagram





$g^{(1)} = [1, 1]$
$g^{(2)} = [1, 0]$

$n(L+m) = 2(5+1) = 12$ columns

$C = [11, 10, 11, 01, 01, 10]$

$$[G] = \begin{bmatrix} 11 & 10 & 00 & 00 & 00 & 00 \\ 00 & 11 & 10 & 00 & 00 & 00 \\ 00 & 00 & 11 & 10 & 00 & 00 \\ 00 & 00 & 00 & 11 & 10 & 00 \\ 00 & 00 & 00 & 00 & 11 & 10 \end{bmatrix}$$

$$C = dG$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 11 & 10 & 00 & 00 & 00 & 00 \\ 00 & 11 & 10 & 00 & 00 & 00 \\ 00 & 00 & 11 & 10 & 00 & 00 \\ 00 & 00 & 00 & 11 & 10 & 00 \\ 00 & 00 & 00 & 00 & 00 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 10 & 11 & 01 & 01 & 10 \end{bmatrix}$$